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Hierarchical Coding to Enable Scalability and Flexibility in Heterogeneous Cloud Storage

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Outline

Introduction

- Latency-reliability trade-off in storage systems
- Heterogeneity, scalability, and flexibility

Preliminaries

- Existing literature
- Cauchy Reed-Solomon (CRS) codes

Constructions

- Double-level codes
- Hierarchical codes
- Properties

Conclusion

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Storage Systems in the Age of Big Data

- Data-intensive applications push forward the innovation of storage systems
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 - In-memory analytics
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 - Cloud storage
 - Persistent memory, computational storage



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 - Cloud storage
 - Persistent memory, computational storage
- Distributed storage (we focus here on cloud storage)
 - Low latency



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 - Occurrence of a large number of errors is rare
- Codes with hierarchical locality provides a trade-off between high reliability and low latency

Heterogeneity

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Expand the backbone network to accommodate additional workload, without rebuilding the entire infrastructure

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Expand the backbone network to accommodate additional workload, without rebuilding the entire infrastructure

Flexibility

- The usage rate of a piece of data is not likely to remain unchanged in dynamic cloud storage
 - Cold data become hot, hot data become cold

Heterogeneity

Allows non-identical local data lengths and unequal local protection

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Scalability

Enables adding a new local cloud without changing the encoding-decoding components (in the generator matrix) of the already-existing local clouds.

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Enables adding a new local cloud without changing the encoding-decoding components (in the generator matrix) of the already-existing local clouds.

Flexibility

Enables dynamic split of a local cloud into smaller clouds without worsening the global ECC capability nor changing the remaining components

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- Generalized integrated interleaved (GII) Codes [1][2]
 - Support a large set of error patterns
 - Distribution of the data symbols is highly restricted
 - Local codewords are equally protected

Yingquan Wu. "Generalized integrated interleaved codes". *IEEE Transactions on Information Theory 63.2 (Nov. 2017), pp. 1102–1119.* Xinmiao Zhang. "Generalized Three-Layer Integrated Interleaved Codes". *IEEE Communications Letters 22.3 (2018), pp. 442–445.*

- Generalized integrated interleaved (GII) Codes ^{[1][2]}
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- Extended integrated interleaved (EII) Codes ^[3]

No hierarchical solution is provided

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Sum-rank codes ^[4]

- Maximal recoverability and flexibility
- Field size exponential in the maximum codeword length

^[1] Yingquan Wu. "Generalized integrated interleaved codes". IEEE Transactions on Information Theory 63.2 (Nov. 2017), pp. 1102–1119.

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^[3] Mario Blaum and Steven R Hetzler. "Extended Product and Integrated Interleaved Codes". IEEE Trans. Inf. Theory 64.3 (2018), pp. 1497–1513.

^[4] Umberto Martnez-Penas and Frank R Kschischang. "Universal and dynamic locally repairable codes with maximal recoverability via sum-rank codes". 2018 56th Annual Allerton Conference on Communication, Control, and Computing (Allerton). IEEE. 2018, pp. 792–799.

CRS-based parity-check matrix with minimum distance (t+1)

$$\mathbf{M} = \begin{bmatrix} \frac{1}{a_1 - b_1} & \frac{1}{a_1 - b_2} & \cdots & \frac{1}{a_1 - b_r} & \cdots & \frac{1}{a_1 - b_t} \\ \frac{1}{a_2 - b_1} & \frac{1}{a_2 - b_2} & \cdots & \frac{1}{a_2 - b_r} & \cdots & \frac{1}{a_2 - b_t} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{1}{a_s - b_1} & \frac{1}{a_s - b_2} & \cdots & \frac{1}{a_s - b_r} & \cdots & \frac{1}{a_s - b_t} \\ -1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & \cdots & 0 \end{bmatrix}^{\mathsf{T}}$$

CRS-based parity-check matrix with minimum distance (t+1)



> Cauchy matrix of size $s \times t$

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Regative identity matrix

- > Cauchy matrix of size $s \times t$
- > Negative identity matrix of size $r \times r$

CRS-based parity-check matrix with minimum distance (t+1)

$$\mathbf{M} = \begin{bmatrix} \frac{1}{a_1 - b_1} & \frac{1}{a_1 - b_2} & \cdots & \frac{1}{a_1 - b_r} & \cdots & \frac{1}{a_1 - b_t} \\ \frac{1}{a_2 - b_1} & \frac{1}{a_2 - b_2} & \cdots & \frac{1}{a_2 - b_r} & \cdots & \frac{1}{a_2 - b_t} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{1}{a_s - b_1} & \frac{1}{a_s - b_2} & \cdots & \frac{1}{a_s - b_r} & \cdots & \frac{1}{a_s - b_t} \\ -1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & \cdots & 0 \end{bmatrix}^{\mathsf{T}}.$$

Zero matrix

- > Cauchy matrix of size $s \times t$
- > Negative identity matrix of size $r \times r$
- > Zero matrix of size $r \times (t r)$

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Construction of double-level accessible codes based on CRS codes

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_{k_1} & \mathbf{A}_{1,1} & \mathbf{0} & \mathbf{A}_{1,2} & \dots & \mathbf{0} & \mathbf{A}_{1,p} \\ \hline \mathbf{0} & \mathbf{A}_{2,1} & \mathbf{I}_{k_2} & \mathbf{A}_{2,2} & \dots & \mathbf{0} & \mathbf{A}_{2,p} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \mathbf{0} & \mathbf{A}_{p,1} & \mathbf{0} & \mathbf{A}_{p,2} & \dots & \mathbf{I}_{k_p} & \mathbf{A}_{p,p} \end{bmatrix}$$

$$\mathbf{T}_x = \begin{bmatrix} \mathbf{A}_{x,x} & \mathbf{B}_{x,1} & \dots & \mathbf{B}_{x,p} \\ \hline \mathbf{U}_x & \mathbf{Z}_x \end{bmatrix}$$

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Construction of double-level accessible codes based on CRS codes

$$\mathbf{I}_{k_{x}}:k_{x} \times k_{x} \\ \mathbf{A}_{x,x}:k_{x} \times r_{x} \\ \mathbf{A}_{x,y}:k_{x} \times \delta_{y} \\ \mathbf{B}_{x,y}:k_{x} \times \delta_{y} \\ \mathbf{G} = \begin{bmatrix} \mathbf{I}_{k_{1}} & \mathbf{A}_{1,1} & \mathbf{0} & \mathbf{A}_{1,2} & \dots & \mathbf{0} & \mathbf{A}_{1,p} \\ \hline \mathbf{0} & \mathbf{A}_{2,1} & \mathbf{I}_{k_{2}} & \mathbf{A}_{2,2} & \dots & \mathbf{0} & \mathbf{A}_{2,p} \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \mathbf{0} & \mathbf{A}_{p,1} & \mathbf{0} & \mathbf{A}_{p,2} & \dots & \mathbf{I}_{k_{p}} & \mathbf{A}_{p,p} \end{bmatrix} \\ \mathbf{A}_{x,y}:k_{x} \times r_{y}; \mathbf{A}_{x,y} = \mathbf{B}_{x,y} \mathbf{U}_{y}$$

Erasure correction capability

> 1st layer error correction capability

•
$$d_{1,x} = r_x - \delta_x + 1$$

> 2nd layer error correction capability

•
$$d_{2,x,i} = r_x - \delta_x + \delta + 1$$

• $\delta = \delta_1 + \dots + \delta_p$

Construction of double-level accessible codes based on CRS codes

$$\mathbf{I}_{k_{x}}:k_{x} \times k_{x} \\ \mathbf{A}_{x,x}:k_{x} \times r_{x} \\ \mathbf{A}_{x,x}:k_{x} \times r_{x} \\ \mathbf{B}_{x,y}:k_{x} \times \delta_{y} \\ \mathbf{U}_{x}:\delta_{x} \times r_{x} \\ \mathbf{A}_{x,y}:k_{x} \times r_{y}; \mathbf{A}_{x,y} = \mathbf{B}_{x,y}\mathbf{U}_{y} \\ \mathbf{U}_{y} = \mathbf{A}_{x,y}\mathbf{U}_{x} + \mathbf{A}_{x,y} = \mathbf{B}_{x,y}\mathbf{U}_{y}$$

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Double-Level Codes

Construction of double-level accessible codes based on CRS codes

 $\succ \mathbf{I}_{k_x}: k_x \times k_x$ $\mathbf{A}_{1,1}$ $\mathbf{A}_{1,2} \mid \ldots \mid$ 0 $\mathbf{A}_{1,p}$ \mathbf{I}_{k_1} $\succ \mathbf{A}_{x.x}: k_x \times r_x$ \mathbf{I}_{k_2} $\mathbf{A}_{2,1}$ $\mathbf{A}_{2,2} \mid$ 0 0 $\mathbf{A}_{2,p}$ $\succ \mathbf{B}_{x,y}: k_x \times \delta_y \quad \mathbf{G} = \Big|$ $\succ \mathbf{U}_{\mathbf{x}}: \delta_{\mathbf{x}} \times r_{\mathbf{x}}$ $\mathbf{0} \mid \mathbf{A}_{p,2}$ $\mathbf{A}_{p,1}$ $oxed{\mathbf{I}}_{k_n} \mid \mathbf{A}_{p,p}$ 0 $\succ \mathbf{A}_{x,v}: k_x \times r_v; \mathbf{A}_{x,v} = \mathbf{B}_{x,v} \mathbf{U}_v$

Erasure correction capability

- > 1st layer error correction capability
 - $d_{1,x} = r_x \delta_x + 1$
- 2nd layer error correction capability

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$$d_{2,x,i} = r_x - \delta_x + \delta + 1$$

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Erasure correction capability

> 1st layer error correction capability

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$$d_{1,x} = r_x - \delta_x + 1$$

- > 2nd layer error correction capability
 - $d_{2,x,i} = r_x \delta_x + \delta + 1$
 - $\delta = \delta_1 + \dots + \delta_p$



Double-level codes

$$\mathbf{T}_{1} = \mathbf{T}_{2} = \begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \mathbf{U}_{1} & \mathbf{Z}_{1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \mathbf{U}_{2} & \mathbf{Z}_{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta^{2} - \beta^{8}} & \frac{1}{\beta^{2} - \beta^{9}} & \frac{1}{\beta^{2} - \beta^{9}} & \frac{1}{\beta^{2} - \beta^{10}} \\ \frac{1}{\beta^{7} - \beta^{8}} & \frac{1}{\beta^{7} - \beta^{9}} & \frac{1}{\beta^{7} - \beta^{10}} \\ \frac{1}{\beta^{7} - \beta^{8}} & \frac{1}{\beta^{7} - \beta^{9}} & \frac{1}{\beta^{7} - \beta^{10}} \\ \frac{1}{\beta^{7} - \beta^{8}} & \frac{1}{\beta^{7} - \beta^{9}} & \frac{1}{\beta^{7} - \beta^{10}} \\ \frac{1}{\beta^{7} - \beta^{8}} & \frac{1}{\beta^{7} - \beta^{9}} & \frac{1}{\beta^{7} - \beta^{10}} \\ \frac{1}{\beta^{7} - \beta^{10}} & \frac{1}{\beta^{7} - \beta^{11}} \\ \frac{1}{\beta^{7} - \beta^{8}} & \frac{1}{\beta^{7} - \beta^{9}} & \frac{1}{\beta^{7} - \beta^{10}} \\ \frac{1}{\beta^{7} - \beta^{10}} & \frac{1}{\beta^{7} - \beta^{11}} \\ \frac{1}{\beta^{7} - \beta^{8}} & \frac{1}{\beta^{7} - \beta^{9}} & \frac{1}{\beta^{7} - \beta^{10}} \\ \frac{1}{\beta^{7} - \beta^{10}} & \frac{1}{\beta^{7} - \beta^{11}} \\ \frac{1}{\beta^{7} - \beta^{11}} & \frac{1}{\beta^{7} - \beta^{11}} \\$$

Local parities

$$\mathbf{T}_{1} = \mathbf{T}_{2} = \begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \mathbf{U}_{1} & \mathbf{Z}_{1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \mathbf{U}_{2} & \mathbf{Z}_{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta^{2} - \beta^{8}} & \frac{1}{\beta^{2} - \beta^{9}} & \frac{1}{\beta^{2} - \beta^{10}} \\ \frac{1}{\beta^{7} - \beta^{8}} & \frac{1}{\beta^{7} - \beta^{9}} & \frac{1}{\beta^{7} - \beta^{10}} \\ \frac{1}{\beta^{7} - \beta^{10}} & \frac{1}{\beta^{7} - \beta^{11}} \end{bmatrix} = \begin{bmatrix} \frac{\beta^{5} - \beta^{12} - \beta^{7}}{\beta^{7}} & \beta^{9} \\ \frac{\beta^{6} - \beta^{12} - \beta^{7}}{\beta^{7} - \beta^{8} - \beta^{12} - \beta^{10}} & \frac{\beta^{7} - \beta^{11}}{\beta^{7} - \beta^{10}} \\ \frac{\beta^{7} - \beta^{11} - \beta^{11}}{\beta^{7} - \beta^{8} - \beta^{11} - \beta^{11}} & \frac{\beta^{7} - \beta^{11}}{\beta^{7} - \beta^{11}} \end{bmatrix} = \begin{bmatrix} \beta^{5} - \beta^{12} - \beta^{7} & \beta^{9} \\ \frac{\beta^{7} - \beta^{7} - \beta^{7} - \beta^{7} - \beta^{7} - \beta^{7} - \beta^{7} - \beta^{11} - \beta^{11} \\ \frac{\beta^{7} - \beta^{8} - \beta^{8} - \beta^{12} - \beta^{8} - \beta^{12} - \beta^{11} - \beta^{11} - \beta^{11} \\ \frac{\beta^{7} - \beta^{8} - \beta^{11} \\ \frac{\beta^{7} - \beta^{8} - \beta^{11} \\ \frac{\beta^{7} - \beta^{8} - \beta^{11} - \beta^{11}$$

Cross parities

$$\mathbf{T}_{1} = \mathbf{T}_{2} = \begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \mathbf{U}_{1} & \mathbf{Z}_{1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \mathbf{U}_{2} & \mathbf{Z}_{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta^{2} - \beta^{8}} & \frac{1}{\beta^{2} - \beta^{9}} & \frac{1}{\beta^{2} - \beta^{9}} & \frac{1}{\beta^{2} - \beta^{10}} \\ \frac{1}{\beta^{2} - \beta^{8}} & \frac{1}{\beta^{7} - \beta^{9}} & \frac{1}{\beta^{7} - \beta^{10}} & \frac{1}{\beta^{3} - \beta^{11}} \\ \frac{1}{\beta^{2} - \beta^{11}} & \frac{1}{\beta^{7} - \beta^{9}} & \frac{1}{\beta^{7} - \beta^{10}} & \frac{1}{\beta^{7} - \beta^{11}} \\ \frac{1}{\beta^{7} - \beta^{8}} & \frac{1}{\beta^{7} - \beta^{9}} & \frac{1}{\beta^{7} - \beta^{10}} & \frac{1}{\beta^{7} - \beta^{11}} \\ \frac{1}{\beta^{7} - \beta^{8}} & \frac{1}{\beta^{7} - \beta^{9}} & \frac{1}{\beta^{7} - \beta^{10}} & \frac{1}{\beta^{7} - \beta^{11}} \\ \frac{1}{\beta^{7} - \beta^{8}} & \frac{1}{\beta^{7} - \beta^{9}} & \frac{1}{\beta^{7} - \beta^{10}} & \frac{1}{\beta^{7} - \beta^{11}} \\ \frac{1}{\beta^{7} - \beta^{8}} & \frac{1}{\beta^{7} - \beta^{10}} & \frac{1}{\beta^{7} - \beta^{10}} & \frac{1}{\beta^{7} - \beta^{11}} \\ \frac{1}{\beta^{7} - \beta^{8}} & \frac{1}{\beta^{7} - \beta^{10}} & \frac{1}{\beta^{7} - \beta^{10}} & \frac{1}{\beta^{7} - \beta^{11}} \\ \frac{1}{\beta^{7} - \beta^{8}} & \frac{1}{\beta^{7} - \beta^{10}} & \frac{1}{\beta^{7} - \beta^{10}} & \frac{1}{\beta^{7} - \beta^{11}} \\ \frac{1}{\beta^{7} - \beta^{8}} & \frac{1}{\beta^{7} - \beta^{10}} & \frac{1}{\beta^{7} - \beta^{10}} & \frac{1}{\beta^{7} - \beta^{11}} \\ \frac{1}{\beta^{7} - \beta^{11}} & \frac{1}{\beta^{7} - \beta^{11}} & \frac{1}{\beta^{7} - \beta^{11}} \\ \frac{1}{\beta^{7} - \beta^{11}} & \frac{1}{\beta^{7} - \beta^{11}} & \frac{1}{\beta^{7} - \beta^{11}} \\ \frac{1}{\beta^{7} - \beta^{11}} & \frac{1}{\beta^{7} - \beta^{11}} & \frac{1}{\beta^{7} - \beta^{11}} \\ \frac{1}{\beta^{7} - \beta^{11}} & \frac{1}{\beta^{7} - \beta^{11}} & \frac{1}{\beta^{7} - \beta^{11}} \\ \frac{1}{\beta^{7} - \beta^{11}} & \frac{1}{\beta^{7} - \beta^{11}} & \frac{1}{\beta^{7} - \beta^{11}} \\ \frac{1}{\beta^{7} - \beta^{11} - \beta^{11}} & \frac{1}{\beta^{7} - \beta^{11}} & \frac{1}{\beta^{7} - \beta^{11}} \\ \frac{1}{\beta^{7} - \beta^{11} - \beta^{11} - \beta^{11}} & \frac{1}{\beta^{7} - \beta^{11}} & \frac{1}{\beta^{7} - \beta^{11}} \\ \frac{1}{\beta^{7} - \beta^{11} - \beta^{$$

Messages and codewords

$$\begin{array}{c} (1,\beta,\beta^{2}, \quad \beta^{14},0,0) \quad (\beta,1,0, \quad \beta^{6},0,\beta^{13}) \\ (1,\beta,\beta^{2}) \quad \left[\begin{array}{c|c|c} 1 & 0 & 0 & \beta^{5} & \beta^{12} & \beta^{7} & 0 & 0 & 0 & \beta^{13} & \beta^{9} & \beta^{3} \\ 0 & 1 & 0 & 1 & \beta^{4} & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^{6} & 1 \\ 0 & 0 & 1 & \beta^{2} & \beta^{14} & \beta^{3} & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^{4} \\ \hline 0 & 0 & 0 & \beta^{13} & \beta^{9} & \beta^{3} & 1 & 0 & 0 & \beta^{5} & \beta^{12} & \beta^{7} \\ \hline 0 & 0 & 0 & \beta^{10} & \beta^{6} & 1 & 0 & 1 & 0 & 1 & \beta^{4} & \beta^{11} \\ \hline 0 & 0 & 0 & \beta^{10} & \beta^{6} & 1 & 0 & 1 & 0 & 1 & \beta^{4} & \beta^{11} \\ \hline 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^{4} & 0 & 0 & 1 & \beta^{2} & \beta^{14} & \beta^{3} \end{array} \right] \\ \mathbf{T}_{1} = \mathbf{T}_{2} = \left[\begin{array}{c|c} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \hline \mathbf{U}_{1} & \mathbf{Z}_{1} \end{array} \right] = \left[\begin{array}{c|c} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \hline \mathbf{U}_{2} & \mathbf{Z}_{2} \end{array} \right] = \left[\begin{array}{c|c} \frac{\frac{1}{\beta^{2} - \beta^{8}} & \frac{1}{\beta^{2} - \beta^{9}} & \frac{1}{\beta^{2} - \beta^{10}} & \frac{1}{\beta^{2} - \beta^{11}} \\ \frac{\beta^{3} - \beta^{8} & \beta^{3} - \beta^{9} & \beta^{3} - \beta^{10}} \\ \frac{\beta^{3} - \beta^{8} & \beta^{3} - \beta^{9} & \beta^{3} - \beta^{10}} \\ \frac{\beta^{3} - \beta^{8} & \beta^{3} - \beta^{9} & \beta^{3} - \beta^{10}} \\ \frac{\beta^{3} - \beta^{8} & \beta^{3} - \beta^{9} & \beta^{3} - \beta^{10}} \\ \frac{\beta^{3} - \beta^{11} & \beta^{7} - \beta^{10}} \\ \frac{\beta^{4} & 1 & \beta^{9} & \beta^{7} \end{array} \right] = \left[\begin{array}{c} \beta^{5} & \beta^{12} & \beta^{7} & \beta^{9} \\ \frac{\beta^{2} & \beta^{14} & \beta^{11} & \beta^{6} \\ \frac{\beta^{2} & \beta^{14} & \beta^{3} & \beta^{10} \\ \frac{\beta^{2} & \beta^{14} & \beta^{3} & \beta^{10} \\ \frac{\beta^{3} - \beta^{8} & \beta^{3} - \beta^{9} & \beta^{3} - \beta^{10} & \beta^{3} - \beta^{11} \\ \frac{\beta^{3} - \beta^{11} & \beta^{7} - \beta^{10} & \frac{\beta^{3} - \beta^{11} & \beta^{1} & \beta^{1} \\ \frac{\beta^{4} & 1 & \beta^{9} & \beta^{7} \end{array} \right]$$

Local erasure correction

	$(1, \ \beta^2, \)$, 0, 0)	(eta,1,0,	$eta^6,0,eta^{13})$	
$(1,eta,eta^2)$	$\begin{bmatrix} 1 & 0 & 0 & \beta^5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \beta^2 \\ \hline 0 & 0 & 0 & \beta^{13} \end{bmatrix}$	$\begin{array}{ccc} \beta^{12} & \beta^7 \\ \beta^4 & \beta^{11} \\ \beta^{14} & \beta^3 \\ \hline \rho^9 & \rho^3 \end{array}$	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	
(eta,1,0)	$\left[\begin{array}{ccccc} 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & \beta^{10} \\ 0 & 0 & 0 & \beta^{14} \end{array}\right]$	$egin{array}{ccc} eta & eta \ eta^6 & 1 \ eta^{10} & eta^4 \end{array}$	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	$egin{array}{cccccccccccccccccccccccccccccccccccc$	
$\mathbf{T}_1 = \mathbf{T}_2 = \begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \hline \mathbf{U}_1 & \mathbf{Z}_1 \end{bmatrix}$	$= egin{bmatrix} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \ \hline \mathbf{U}_2 & \mathbf{Z}_2 \end{bmatrix} =$	$\begin{bmatrix} \frac{1}{\beta - \beta^8} \\ \frac{1}{\beta^2 - \beta^8} \\ \frac{1}{\beta^3 - \beta^8} \\ \frac{1}{\beta^7 - \beta^8} \end{bmatrix}$	$ \frac{1}{\beta - \beta^9} \qquad \frac{1}{\beta - $	$ \begin{bmatrix} \frac{1}{\beta^{10}} & \frac{1}{\beta - \beta^{11}} \\ \frac{1}{\beta^{10}} & \frac{1}{\beta^2 - \beta^{11}} \\ \frac{1}{\beta^{10}} & \frac{1}{\beta^3 - \beta^{11}} \\ \frac{1}{\beta^{10}} & \frac{1}{\beta^7 - \beta^{11}} \end{bmatrix} = $	$\begin{bmatrix} \beta^5 & \beta^{12} & \beta^7 & \beta^9 \\ 1 & \beta^4 & \beta^{11} & \beta^6 \\ \beta^2 & \beta^{14} & \beta^3 & \beta^{10} \\ \hline \beta^4 & 1 & \beta^9 & \beta^7 \end{bmatrix}$

$$\mathbf{T}_{1} = \mathbf{T}_{2} = \begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \hline \mathbf{U}_{1} & \mathbf{Z}_{1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \hline \mathbf{U}_{2} & \mathbf{Z}_{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta - \beta^{8}} & \frac{1}{\beta - \beta^{9}} & \frac{1}{\beta - \beta^{10}} & \frac{1}{\beta - \beta^{11}} \\ \frac{1}{\beta^{2} - \beta^{8}} & \frac{1}{\beta^{2} - \beta^{9}} & \frac{1}{\beta^{2} - \beta^{10}} & \frac{1}{\beta^{2} - \beta^{11}} \\ \frac{1}{\beta^{3} - \beta^{8}} & \frac{1}{\beta^{3} - \beta^{9}} & \frac{1}{\beta^{3} - \beta^{10}} & \frac{1}{\beta^{3} - \beta^{11}} \\ \frac{1}{\beta^{7} - \beta^{8}} & \frac{1}{\beta^{7} - \beta^{9}} & \frac{1}{\beta^{7} - \beta^{10}} & \frac{1}{\beta^{7} - \beta^{11}} \end{bmatrix} = \begin{bmatrix} \beta^{5} & \beta^{12} & \beta^{7} & \beta^{9} \\ 1 & \beta^{4} & \beta^{11} & \beta^{6} \\ \frac{\beta^{2} & \beta^{14} & \beta^{3} & \beta^{10} \\ \frac{\beta^{4} & 1 & \beta^{9} & \beta^{7} \end{bmatrix}$$

$$(1, e_1, \beta^2, e_2, 0, 0, e_3) \begin{bmatrix} \beta^5 & \beta^{12} & \beta^7 \\ 1 & \beta^4 & \beta^{11} \\ \beta^2 & \beta^{14} & \beta^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \beta^4 & 1 & \beta^9 \end{bmatrix}$$

$$(1,\beta,\beta^{2}, 0,0) \quad (\beta,1,0, \beta^{6},0,\beta^{13})$$

$$(1,\beta,\beta^{2}, 0,0) \quad (\beta,1,0, \beta^{6},0,\beta^{13})$$

$$(1,\beta,\beta^{2}) \quad \begin{bmatrix} 1 & 0 & 0 & \beta^{5} & \beta^{12} & \beta^{7} & 0 & 0 & 0 & \beta^{13} & \beta^{9} & \beta^{3} \\ 0 & 1 & 0 & 1 & \beta^{4} & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^{6} & 1 \\ 0 & 0 & 1 & \beta^{2} & \beta^{14} & \beta^{3} & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^{4} \\ \hline 0 & 0 & 0 & \beta^{13} & \beta^{9} & \beta^{3} & 1 & 0 & 0 & \beta^{5} & \beta^{12} & \beta^{7} \\ 0 & 0 & 0 & \beta^{10} & \beta^{6} & 1 & 0 & 1 & 0 & 1 & \beta^{4} & \beta^{11} \\ 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^{4} & 0 & 0 & 1 & \beta^{2} & \beta^{14} & \beta^{3} \end{bmatrix}$$

$$\mathbf{T}_{1} = \mathbf{T}_{2} = \begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \hline \mathbf{U}_{1} & \mathbf{Z}_{1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \hline \mathbf{U}_{2} & \mathbf{Z}_{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta - \beta^{8}} & \frac{1}{\beta - \beta^{9}} & \frac{1}{\beta - \beta^{10}} & \frac{1}{\beta - \beta^{11}} \\ \frac{1}{\beta^{2} - \beta^{8}} & \frac{1}{\beta^{2} - \beta^{9}} & \frac{1}{\beta^{2} - \beta^{10}} & \frac{1}{\beta^{2} - \beta^{11}} \\ \frac{1}{\beta^{3} - \beta^{8}} & \frac{1}{\beta^{3} - \beta^{9}} & \frac{1}{\beta^{3} - \beta^{10}} & \frac{1}{\beta^{3} - \beta^{11}} \\ \frac{1}{\beta^{7} - \beta^{8}} & \frac{1}{\beta^{7} - \beta^{9}} & \frac{1}{\beta^{7} - \beta^{10}} & \frac{1}{\beta^{7} - \beta^{11}} \\ \end{bmatrix} = \begin{bmatrix} \beta^{5} & \beta^{12} & \beta^{7} & \beta^{9} \\ 1 & \beta^{4} & \beta^{11} & \beta^{6} \\ \frac{\beta^{2} & \beta^{14} & \beta^{3} & \beta^{10} \\ \frac{\beta^{4} & 1 & \beta^{9} & \beta^{7} \end{bmatrix}$$

$$(1, e_1, \beta^2, e_2, 0, 0, e_3) \begin{bmatrix} \beta^5 & \beta^{12} & \beta^7 \\ 1 & \beta^4 & \beta^{11} \\ \beta^2 & \beta^{14} & \beta^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \beta^4 & 1 & \beta^9 \end{bmatrix}$$

$$\mathbf{T}_{1} = \mathbf{T}_{2} = \begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \hline \mathbf{U}_{1} & \mathbf{Z}_{1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \hline \mathbf{U}_{2} & \mathbf{Z}_{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta^{2} - \beta^{8}} & \frac{1}{\beta^{2} - \beta^{9}} & \frac{1}{\beta^{2} - \beta^{10}} & \frac{1}{\beta^{2} - \beta^{10}} \\ \frac{1}{\beta^{3} - \beta^{8}} & \frac{1}{\beta^{3} - \beta^{9}} & \frac{1}{\beta^{3} - \beta^{10}} & \frac{1}{\beta^{3} - \beta^{11}} \\ \hline \frac{1}{\beta^{7} - \beta^{8}} & \frac{1}{\beta^{7} - \beta^{9}} & \frac{1}{\beta^{7} - \beta^{10}} & \frac{1}{\beta^{7} - \beta^{11}} \end{bmatrix} = \begin{bmatrix} 1 & \beta^{4} & \beta^{11} & \beta^{6} \\ \beta^{2} & \beta^{14} & \beta^{3} & \beta^{10} \\ \hline \beta^{4} & 1 & \beta^{9} & \beta^{7} \end{bmatrix}$$

$$(1, e_1, \beta^2, e_2, 0, 0, e_3) \begin{bmatrix} \beta^5 & \beta^{12} & \beta^7 \\ 1 & \beta^4 & \beta^{11} \\ \beta^2 & \beta^{14} & \beta^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \beta^4 & 1 & \beta^9 \end{bmatrix}$$

$$(1,\beta,\beta^{2}, \beta^{14},0,0) \quad (\beta,1,0, \beta^{6},0,\beta^{13})$$

$$(1,\beta,\beta^{2}) \begin{bmatrix} 1 & 0 & 0 & \beta^{5} & \beta^{12} & \beta^{7} & 0 & 0 & 0 & \beta^{13} & \beta^{9} & \beta^{3} \\ 0 & 1 & 0 & 1 & \beta^{4} & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^{6} & 1 \\ 0 & 0 & 1 & \beta^{2} & \beta^{14} & \beta^{3} & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^{4} \\ \hline 0 & 0 & 0 & \beta^{13} & \beta^{9} & \beta^{3} & 1 & 0 & 0 & \beta^{5} & \beta^{12} & \beta^{7} \\ 0 & 0 & 0 & \beta^{10} & \beta^{6} & 1 & 0 & 1 & 0 & 1 & \beta^{4} & \beta^{11} \\ 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^{4} & 0 & 0 & 1 & \beta^{2} & \beta^{14} & \beta^{3} \end{bmatrix}$$

$$\mathbf{T}_{1} = \mathbf{T}_{2} = \begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \hline \mathbf{U}_{1} & \mathbf{Z}_{1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \hline \mathbf{U}_{2} & \mathbf{Z}_{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta - \beta^{8}} & \frac{1}{\beta - \beta^{9}} & \frac{1}{\beta - \beta^{10}} & \frac{1}{\beta - \beta^{10}} \\ \frac{1}{\beta^{2} - \beta^{8}} & \frac{1}{\beta^{2} - \beta^{9}} & \frac{1}{\beta^{2} - \beta^{10}} & \frac{1}{\beta^{2} - \beta^{11}} \\ \frac{1}{\beta^{3} - \beta^{8}} & \frac{1}{\beta^{3} - \beta^{9}} & \frac{1}{\beta^{3} - \beta^{10}} & \frac{1}{\beta^{3} - \beta^{11}} \\ \hline \frac{1}{\beta^{7} - \beta^{8}} & \frac{1}{\beta^{7} - \beta^{9}} & \frac{1}{\beta^{7} - \beta^{10}} & \frac{1}{\beta^{7} - \beta^{11}} \\ \hline \frac{1}{\beta^{4}} & 1 & \beta^{9} & \beta^{7} \end{bmatrix} = \begin{bmatrix} \beta^{5} & \beta^{12} & \beta^{7} & \beta^{9} \\ 1 & \beta^{4} & \beta^{11} & \beta^{6} \\ \frac{\beta^{2} & \beta^{14} & \beta^{3} & \beta^{10} \\ \hline \beta^{4} & 1 & \beta^{9} & \beta^{7} \end{bmatrix}$$

$$(1, e_1, \beta^2, e_2, 0, 0, e_3) \begin{bmatrix} \beta^5 & \beta^{12} & \beta^7 \\ 1 & \beta^4 & \beta^{11} \\ \beta^2 & \beta^{14} & \beta^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \beta^4 & 1 & \beta^9 \end{bmatrix}$$
$$(e_1, e_2, e_3) = (\beta, \beta^{14}, \beta^7)$$

Global erasure correction

	(,	β^2 ,	,0)	(eta,1,0,	$eta^6,0,\mu$	$3^{13})$		
$(1,eta,eta^2)$	$ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} $	$\begin{array}{c c}0 & \beta^5 \\ 0 & 1 \\ 1 & \beta^2 \end{array}$	$\begin{array}{ccc} \beta^{12} & \beta^7 \\ \beta^4 & \beta^{11} \\ \beta^{14} & \beta^3 \end{array}$	$\begin{array}{cccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$	$ \begin{array}{c c} \beta^{13} & \beta^9 \\ \beta^{10} & \beta^6 \\ \beta^{14} & \beta^{10} \end{array} $	$\left[\begin{array}{c} \beta^3 \\ 1 \\ \beta^4 \end{array} \right]$		
(eta,1,0)	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{array}{c c}0&\beta^{13}\\0&\beta^{10}\\0&\beta^{14}\end{array}$	$egin{array}{ccc} eta^9 & eta^3 \ eta^6 & 1 \ eta^{10} & eta^4 \end{array}$	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{vmatrix} \beta^5 & \beta^{12} \\ 1 & \beta^4 \\ \beta^2 & \beta^{14} \end{vmatrix}$	$\left[egin{smallmatrix} eta^{\prime} \ eta^{11} \ eta^{3} \end{array} ight]$		
$\mathbf{T}_1 = \mathbf{T}_2 = \left[egin{array}{c c} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \ \hline \mathbf{U}_1 & \mathbf{Z}_1 \end{array} ight]$	$= \left[egin{array}{c} \mathbf{A}_{2,2} \ \hline \mathbf{U}_2 \end{array} ight]$	$ig rac{\mathbf{B}_{2,1}}{\mathbf{Z}_2} ig] =$	$\begin{bmatrix} \frac{1}{\beta - \beta^8} \\ \frac{1}{\beta^2 - \beta^8} \\ \frac{-\beta^3 - \beta^8}{\beta^3 - \beta^8} \\ \frac{1}{\beta^7 - \beta^8} \end{bmatrix}$	$ \frac{\frac{1}{\beta - \beta^9}}{\frac{1}{\beta^2 - \beta^9}} \qquad \overline{\beta^2} - \frac{\beta^2}{\beta^2} - \frac{\beta^2}{\beta^3} - \frac{\beta^2}{\beta^3} - \frac{\beta^2}{\beta^3} - \frac{\beta^2}{\beta^7} - \frac{\beta^2}{\beta^7$	$\begin{array}{c c c c c c c c c c c c c c c c c c c $	$\begin{bmatrix} \beta^5 \\ 1 \\ \beta^2 \\ \hline \beta^4 \end{bmatrix} = \begin{bmatrix} \beta^5 \\ 1 \\ \beta^2 \\ \hline \beta^4 \end{bmatrix}$	$\frac{\beta^{12}}{\beta^4}$ $\frac{\beta^{14}}{1}$	$egin{array}{c c} eta^7 & eta^9 \ eta^{11} & eta^6 \ eta^3 & eta^{10} \ eta^9 & eta^7 \end{array} \end{bmatrix}$

Global erasure correction

	($,eta^{2},$,0)	(eta,1,0	β^6	$,0,eta^{13})$		
$(1,eta,eta^2)$	$\begin{bmatrix} 1\\0\\0 \end{bmatrix}$	$egin{array}{c c} 0 & 0 & \beta^5 \ 1 & 0 & 1 \ 0 & 1 & \beta^2 \end{array}$	$egin{array}{ccc} eta^{12} & eta^7 \ eta^4 & eta^{11} \ eta^{14} & eta^3 \end{array}$	0 0 0 0 0 0 0 0 0	$\begin{array}{c c}0 & \beta^{13} \\ 0 & \beta^{10} \\ 0 & \beta^{14} \end{array}$	$ \begin{bmatrix} \beta^9 & \beta^3 \\ \beta^6 & 1 \\ \beta^{10} & \beta^4 \end{bmatrix} $		
(eta,1,0)	0 0 0	$\begin{array}{cccc} 0 & 0 & \beta^{13} \\ 0 & 0 & \beta^{10} \\ 0 & 0 & \beta^{14} \end{array}$	$egin{array}{ccc} eta^9 & eta^3 \ eta^6 & 1 \ eta^{10} & eta^4 \end{array}$	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	$egin{array}{c c} 0 & eta^5 \ 0 & 1 \ 1 & eta^2 \end{array}$	$ \begin{array}{ccc} \beta^{12} & \beta^7 \\ \beta^4 & \beta^{11} \\ \beta^{14} & \beta^3 \end{array} \right] $		
$\mathbf{T}_1 = \mathbf{T}_2 = egin{bmatrix} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \ \hline \mathbf{U}_1 & \mathbf{Z}_1 \end{bmatrix}$	$=\left[\begin{array}{c} \mathbf{A}\\ \mathbf{U}\end{array}\right]$	$ \begin{bmatrix} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \mathbf{J}_2 & \mathbf{Z}_2 \end{bmatrix} = $	$= \begin{bmatrix} \frac{1}{\beta - \beta^8} \\ \frac{1}{\beta^2 - \beta^8} \\ \frac{1}{\beta^3 - \beta^8} \\ \frac{1}{\beta^7 - \beta^8} \end{bmatrix}$	$ \frac{\frac{1}{\beta - \beta^9}}{\frac{1}{\beta^2 - \beta^9}} \frac{1}{\beta} \frac{1}{\beta} \frac{1}{\beta^3 - \beta^9} \frac{1}{\beta} \frac{1}{\beta^7 - \beta^9} \frac{1}{\beta} $	$ \begin{array}{c c} \frac{1}{\beta - \beta^{10}} & \overline{\beta} \\ \frac{1}{\beta^2 - \beta^{10}} & \overline{\beta}^3 \\ \frac{1}{\beta^3 - \beta^{10}} & \overline{\beta}^3 \\ \frac{1}{\beta^7 - \beta^{10}} & \overline{\beta} \end{array} $	$\begin{bmatrix} \frac{1}{\beta - \beta^{11}} \\ \frac{1}{2 - \beta^{11}} \\ \frac{1}{3 - \beta^{11}} \\ \frac{1}{7 - \beta^{11}} \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$	$egin{array}{ccc} eta^5 & eta^{12} \ 1 & eta^4 \ eta^2 & eta^{14} \ eta^4 & 1 \end{array}$	$ \begin{array}{c c} \beta^7 & \beta^9 \\ \beta^{11} & \beta^6 \\ \beta^3 & \beta^{10} \\ \hline \beta^9 & \beta^7 \end{array} \right] $

$$(e_1, e_2, \beta^2, e_3, e_4, 0) = (e_1, e_2, \beta^2, e_3', e_4', \beta)$$
$$= (e_1', e_2', \beta^2, e_3', e_4', \beta)$$

Global erasure correction

$$(1, \beta, \beta^{2}) = \begin{pmatrix} \beta^{2}, \beta^$$

 $=(e_1',e_2',\beta^2,e_3',e_4',\beta)$

Global erasure correction

	$(\ \beta^2,$,0) $(eta,1)$	$,0, \qquad eta^6,0,eta^{13})$	
$(1,eta,eta^2)$	$\begin{bmatrix} 1 & 0 & 0 & \beta^5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \beta^2 \end{bmatrix}$	$\begin{array}{c cccc} \beta^{12} & \beta^{7} & 0 & 0 \\ \beta^{4} & \beta^{11} & 0 & 0 \\ \beta^{14} & \beta^{3} & 0 & 0 \end{array}$	$\begin{bmatrix} 0 & \beta^{13} & \beta^{9} & \beta^{3} \\ 0 & \beta^{10} & \beta^{6} & 1 \\ 0 & \beta^{14} & \beta^{10} & \beta^{4} \end{bmatrix}$	
(eta,1,0)	$\begin{bmatrix} 0 & 0 & 0 & \beta^{13} \\ 0 & 0 & 0 & \beta^{10} \\ 0 & 0 & 0 & \beta^{14} \end{bmatrix}$	$egin{array}{c c} eta^9 & eta^3 & 1 & 0 \ eta^6 & 1 & 0 & 1 \ eta^{10} & eta^4 & 0 & 0 \end{array}$	$\begin{bmatrix} 0 & \beta^5 & \beta^{12} & \beta^7 \\ 0 & 1 & \beta^4 & \beta^{11} \\ 1 & \beta^2 & \beta^{14} & \beta^3 \end{bmatrix}$	
$\mathbf{T}_1 = \mathbf{T}_2 = \left[egin{array}{c c} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \ \hline \mathbf{U}_1 & \mathbf{Z}_1 \end{array} ight]$	$= egin{bmatrix} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \ \hline \mathbf{U}_2 & \mathbf{Z}_2 \end{bmatrix} =$	$\begin{bmatrix} \frac{1}{\beta-\beta^8} & \frac{1}{\beta-\beta^9} \\ \frac{1}{\beta^2-\beta^8} & \frac{1}{\beta^2-\beta^9} \\ \frac{1}{\beta^3-\beta^8} & \frac{1}{\beta^3-\beta^9} \\ \hline \frac{1}{\beta^7-\beta^8} & \frac{1}{\beta^7-\beta^9} \end{bmatrix}$	$ \begin{array}{c c c} \frac{1}{\beta - \beta^{10}} & \frac{1}{\beta - \beta^{11}} \\ \frac{1}{\beta^2 - \beta^{10}} & \frac{1}{\beta^2 - \beta^{11}} \\ \frac{1}{\beta^3 - \beta^{10}} & \frac{1}{\beta^3 - \beta^{11}} \\ \hline \frac{1}{\beta^7 - \beta^{10}} & \frac{1}{\beta^7 - \beta^{11}} \\ \end{array} \right] = \left[\begin{array}{c} \\ \end{array} \right]$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$
e_1', e_2', β^2 e_2'	$(\beta_3, e_4', \beta) \begin{bmatrix} eta^5 & eta^{12} \ 1 & eta^4 \ eta^2 & eta^{14} \ 1 & 0 \ 0 & 1 \ 0 & 0 \end{bmatrix}$	$ \begin{array}{ccc} \beta^7 & \beta^9 \\ \beta^{11} & \beta^6 \\ \beta^3 & \beta^{10} \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{array} $	$(e_1, e_2, \beta^2, e_3) - (0, 0, 0, \beta^{11}, e_2) = (e_1', e_2', \beta^2, e_3')$	$(\beta_{3},e_{4},0),\ (\beta_{3},e_{4}^{\prime},eta),\ ($

Global erasure correction

	$(\ \beta^2,$,0) $(eta,1,0,$	$eta^6,0,eta^{13})$	
$(1,eta,eta^2)$	$\begin{bmatrix} 1 & 0 & 0 & \beta^5 & \beta^1 \\ 0 & 1 & 0 & 1 & \beta^4 \\ 0 & 0 & 1 & \beta^2 & \beta^1 \end{bmatrix}$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	
(eta,1,0)	$\begin{bmatrix} 0 & 0 & 0 & \beta^{13} & \beta^{8} \\ 0 & 0 & 0 & \beta^{10} & \beta^{6} \\ 0 & 0 & 0 & \beta^{14} & \beta^{1} \end{bmatrix}$	$egin{array}{c c c c c c c c c c c c c c c c c c c $	$\begin{bmatrix} \beta^5 & \beta^{12} & \beta^7 \\ 1 & \beta^4 & \beta^{11} \\ \beta^2 & \beta^{14} & \beta^3 \end{bmatrix}$	
$\mathbf{T}_1 = \mathbf{T}_2 = \begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \hline \mathbf{U}_1 & \mathbf{Z}_1 \end{bmatrix}$	$= egin{bmatrix} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \ \hline \mathbf{U}_2 & \mathbf{Z}_2 \end{bmatrix} = egin{bmatrix} - & - \ - & $	$ \frac{\frac{1}{\beta-\beta^8}}{\frac{1}{\beta^2-\beta^8}} \frac{\frac{1}{\beta-\beta^9}}{\frac{\beta^2-\beta^9}{\beta^2-\beta^9}} \frac{\overline{\beta^2-\beta^9}}{\overline{\beta^3-\beta^8}} \frac{\overline{\beta^2-\beta^9}}{\overline{\beta^3-\beta^9}} \frac{\overline{\beta^3-\beta^9}}{\overline{\beta^3-\beta^9}} \frac{\overline{\beta^3-\beta^9}}{\overline{\beta^7-\beta^9}} \frac{\overline{\beta^7-\beta^9}}{\overline{\beta^7-\beta^9}} \frac{\overline{\beta^7-\beta^9}} \frac{\overline{\beta^7-\beta^9}}$	$ \frac{\frac{1}{\beta^{-\beta^{10}}} \left \frac{1}{\beta^{-\beta^{11}}} \right }{\frac{1}{\beta^{2} - \beta^{11}} \left \frac{1}{\beta^{2} - \beta^{11}} \right }{\frac{1}{\beta^{3} - \beta^{11}}} = \begin{bmatrix} \beta^{5} \\ 1 \\ \beta^{2} \\ \frac{\beta^{2}}{\beta^{4}} \\ \frac{1}{\beta^{4}} \\ \beta^{4} \end{bmatrix} $	$ \begin{array}{c cccc} \beta^{12} & \beta^{7} & \beta^{9} \\ \beta^{4} & \beta^{11} & \beta^{6} \\ \beta^{14} & \beta^{3} & \beta^{10} \\ \hline 1 & \beta^{9} & \beta^{7} \end{array} $
$(e_1^\prime,e_2^\prime,eta^2,e_3^\prime)$	$\left(egin{array}{ccc} eta , e_4', eta) \left[egin{array}{ccc} eta ^5 & eta ^{12} & eta \\ 1 & eta ^4 & eta \\ eta ^2 & eta ^{14} & eta \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} ight.$	$ \begin{bmatrix} \beta^7 & \beta^9 \\ \beta^{11} & \beta^6 \\ \beta^3 & \beta^{10} \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} $	$(e_1, e_2, \beta^2, e_3, e_4)$ - $(0, 0, 0, \beta^{11}, \beta^7)$ = $(e'_1, e'_2, \beta^2, e'_3, e'_4)$	$^{(1,0)}_{(4,\beta)}$
$(e_1', e_2', e_3',$	$e_4')=(1,eta,eta^{10},eta^{10})$	7)		

Construction of hierarchical codes based on double-level codes

$$\mathbf{G} = \begin{bmatrix} \mathbf{F}_{1,1} & \mathbf{F}_{1,2} & \dots & \mathbf{F}_{1,p_0} \\ \hline \mathbf{F}_{2,1} & \mathbf{F}_{2,2} & \dots & \mathbf{F}_{2,p_0} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{F}_{p_0,1} & \mathbf{F}_{p_0,2} & \dots & \mathbf{F}_{p_0,p_0} \end{bmatrix}$$

$$\mathbf{T}_{x,i} = egin{bmatrix} \mathbf{A}_{x,x;i,i} & \mathbf{B}_{x,x;i} \mid \mathbf{E}_{x,1;i} \mid \ldots \mid \mathbf{E}_{x,p_0;i} \ \hline \mathbf{U}_{x,i} & \mathbf{Z}_{x,i} \ \hline \mathbf{V}_{x,i} & \mathbf{Z}_{x,i} \end{bmatrix}$$

Construction of hierarchical codes based on double-level codes



Local double-layer codes

UCLA

Construction of hierarchical codes based on double-level codes

- Local double-layer codes
 - $\mathbf{I}_{k_{x,i}}: k_{x,i} \times k_{x,i}, \mathbf{A}_{x,x;i,i}: k_{x,i} \times r_{x,i}$
 - $\mathbf{B}_{x,x;i,j}: k_{x,i} \times \delta_{y,j}, \mathbf{U}_{x,i}: \delta_{x,i} \times r_{x,i}$

•
$$\mathbf{A}_{x,x;i,j} = \mathbf{B}_{x,x;i,j} \mathbf{U}_{x,j}$$

	$\mathbf{F}_{1,1}$	$\mathbf{F}_{1,2}$	•••	\mathbf{F}_{1,p_0}
G	$\mathbf{F}_{2,1}$	$\mathbf{F}_{2,2}$	•••	\mathbf{F}_{2,p_0}
$\mathbf{G} =$:		·	:
	$\mathbf{F}_{p_0,1}$	$\mathbf{F}_{p_0,2}$	•••	\mathbf{F}_{p_0,p_0}



Construction of hierarchical codes based on double-level codes

Local double-layer codes \mathbf{F}_{1,p_0} $\mathbf{F}_{1,1}$ $\mathbf{F}_{1,2}$ $\mathbf{F}_{2,1}$ $\mathbf{F}_{2,\underline{p}_0}$ $\mathbf{F}_{2,2}$ • $\mathbf{I}_{k_{x,i}}: k_{x,i} \times k_{x,i}, \mathbf{A}_{x,x;i,i}: k_{x,i} \times r_{x,i}$ $\mathbf{G} =$ • $\mathbf{B}_{x,x;i,j}: k_{x,i} \times \delta_{y,j}, \mathbf{U}_{x,i}: \delta_{x,i} \times r_{x,i}$ $\mathbf{F}_{p_0,1}$ \mathbf{F}_{p_0,p_0} • $\mathbf{A}_{x,x;i,j} = \mathbf{B}_{x,x;i,j} \mathbf{U}_{x,j}$ Second layer cross parities Generate second layer cross parities 0 $\mathbf{A}_{x,y;1,p_y}$ $\mathbf{A}_{x,y;1,1}$. . . x. (**0** $\mathbf{A}_{x,y;p_x,1}$ $\mathbf{B}_{x,x;i} \mathbf{E}_{x,1;i} \mid$ $\mathbf{E}_{x,p_0;i}$ $\mathbf{A}_{x,x;i,i}$ $\mathbf{T}_{x,i} =$ $\mathbf{Z}_{x,i}$

Construction of hierarchical codes based on double-level codes

- Local double-layer codes
 - $\mathbf{I}_{k_{x,i}}: k_{x,i} \times k_{x,i}, \mathbf{A}_{x,x;i,i}: k_{x,i} \times r_{x,i}$
 - $\mathbf{B}_{x,x:i,i}: k_{x,i} \times \delta_{y,i}, \mathbf{U}_{x,i}: \delta_{x,i} \times r_{x,i}$
 - $\mathbf{A}_{x,x;i,j} = \mathbf{B}_{x,x;i,j} \mathbf{U}_{x,j}$
- Second layer cross parities

 - $\mathbf{A}_{x,y;i,j} = [\mathbf{E}_{x,y;i;j}, \mathbf{E}_{x,y;i;j+1}]\mathbf{V}_{y,j}$





• 1st layer error correction capability > $d_{1,x,i} = r_{x,i} - \delta_{x,i} - 2\gamma_x + 1$

- 2nd layer error correction capability
 *d*_{2,x,i} = r_{x,i} − δ_{x,i} + δ_x + 1
 δ_x = δ_{x,1} + ··· + δ_{x,px}
- 3rd layer error correction capability

$$> d_{3,x,i} = r_{x,i} - \delta_{x,i} + \delta_x - p_x \gamma_x + \gamma + 1$$

$$> \gamma = p_1 \gamma_1 + \dots + p_{p_0} \gamma_{p_0}$$

• Example of a triple-level code

[1	0	0	β^5	eta^{12}	β^7	0	0	0	eta^{13}	β^9	eta^3	0	0	0	eta^3	eta^{12}	eta^{10}	0	0	0	eta^3	eta^{12}	β^{10}]
0	1	0	1	β^4	β^{11}	0	0	0	eta^{10}	β^6	1	0	0	0	β^9	β^3	eta	0	0	0	β^9	β^3	β
0	0	1	β^2	eta^{14}	eta^{3}	0	0	0	eta^{14}	eta^{10}	eta^4	0	0	0	eta^6	1	eta^{13}	0	0	0	eta^6	1	eta^{13}
0	0	0	β^{13}	β^9	β^3	1	0	0	eta^5	β^{12}	β^7	0	0	0	β^3	β^{12}	β^{10}	0	0	0	β^3	β^{12}	β^{10}
0	0	0	β^{10}	β^6	1	0	1	0	1	eta^4	eta^{11}	0	0	0	β^9	β^3	β	0	0	0	β^9	β^3	β
0	0	0	eta^{14}	eta^{10}	eta^4	0	0	1	eta^2	eta^{14}	β^3	0	0	0	eta^6	1	eta^{13}	0	0	0	eta^6	1	β^{13}
0	0	0	β^3	β^{12}	β^{10}	0	0	0	β^3	eta^{12}	β^{10}	1	0	0	β^5	eta^{12}	β^7	0	0	0	β^{13}	β^9	β^3
0	0	0	β^9	eta^3	eta	0	0	0	eta^9	eta^3	eta	0	1	0	1	eta^4	eta^{11}	0	0	0	eta^{10}	eta^6	1
0	0	0	β^6	1	eta^{13}	0	0	0	eta^6	1	eta^{13}	0	0	1	eta^2	eta^{14}	eta^3	0	0	0	eta^{14}	eta^{10}	β^4
0	0	0	β^3	β^{12}	β^{10}	0	0	0	β^3	β^{12}	β^{10}	0	0	0	β^{13}	β^9	β^3	1	0	0	eta^5	β^{12}	β^7
0	0	0	β^9	eta^3	eta	0	0	0	β^9	β^3	β	0	0	0	eta^{10}	eta^6	1	0	1	0	1	β^4	β^{11}
0	0	0	β^6	1	eta^{13}	0	0	0	eta^6	1	eta^{13}	0	0	0	eta^{14}	eta^{10}	eta^4	0	0	1	β^2	eta^{14}	β^3

Local double-level codes

1	0	0	$\beta^{\mathfrak{d}}$	$eta^{{\scriptscriptstyle 12}}$	eta'	0	0	0	β^{13}	eta^{9}	$\beta^{\mathfrak{z}}$	0	0	0	β^3	eta^{12}	eta^{10}	0	0	0	β^3	eta^{12}	eta^{10} -
0	1	0	1	eta^4	eta^{11}	0	0	0	β^{10}	eta^6	1	0	0	0	β^9	eta^{3}	eta	0	0	0	β^9	eta^{3}	eta
0	0	1	β^2	eta^{14}	eta^3	0	0	0	β^{14}	eta^{10}	eta^4	0	0	0	β^6	1	eta^{13}	0	0	0	β^6	1	eta^{13}
0	0	0	β^{13}	β^9	eta^3	1	0	0	β^5	eta^{12}	β^7	0	0	0	β^3	eta^{12}	eta^{10}	0	0	0	β^3	eta^{12}	β^{10}
0	0	0	β^{10}	eta^6	1	0	1	0	1	eta^4	eta^{11}	0	0	0	β^9	eta^{3}	eta	0	0	0	β^9	eta^{3}	eta
0	0	0	β^{14}	eta^{10}	eta^4	0	0	1	β^2	eta^{14}	eta^3	0	0	0	β^6	1	eta^{13}	0	0	0	eta^6	1	eta^{13}
0	0	0	β^3	eta^{12}	eta^{10}	0	0	0	β^3	eta^{12}	eta^{10}	1	0	0	β^5	eta^{12}	β^7	0	0	0	eta^{13}	eta^9	β^3
0	0	0	β^9	eta^{3}	eta	0	0	0	β^9	eta^{3}	eta	0	1	0	1	eta^4	eta^{11}	0	0	0	eta^{10}	eta^6	1
0	0	0	β^6	1	eta^{13}	0	0	0	β^6	1	eta^{13}	0	0	1	β^2	eta^{14}	eta^3	0	0	0	eta^{14}	eta^{10}	eta^{4}
$\left \begin{array}{c} 0 \end{array} \right $	0	0	β^3	eta^{12}	β^{10}	0	0	0	β^3	eta^{12}	β^{10}	0	0	0	β^{13}	β^9	β^3	1	0	0	β^5	β^{12}	β^7
0	0	0	β^9	eta^{3}	eta	0	0	0	β^9	eta^3	eta	0	0	0	β^{10}	eta^6	1	0	1	0	1	eta^4	eta^{11}
					10						10					10	4						

Second layer cross parities

$\begin{bmatrix} 1\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\$	0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0 0 0 0 0 0 0	$\begin{array}{c} \beta^5 \\ 1 \\ \beta^2 \\ \beta^{13} \\ \beta^{10} \\ \beta^{14} \\ \beta^3 \\ \beta^9 \\ \beta^6 \\ \beta^3 \\ \beta^9 \\ \beta^6 \\ \beta^3 \\ \beta^9 \\ \beta^6 \end{array}$	$\beta^{12} \\ \beta^{4} \\ \beta^{14} \\ \beta^{9} \\ \beta^{6} \\ \beta^{10} \\ \beta^{12} \\ \beta^{3} \\ 1 \\ \beta^{12} \\ \beta^{3} \\ 1 \\ 1$	β^{7} β^{11} β^{3} β^{3} 1 β^{4} β^{10} β β^{13} β^{10} β β^{13} β^{13}	0 0 1 0 0 0 0 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0 0 0 0	β^{13} β^{10} β^{14} β^{5} 1 β^{2} β^{3} β^{9} β^{6} β^{3} β^{9} β^{6}	$egin{array}{c} & & & & & & & & & & & & & & & & & & &$	$\begin{array}{c} \beta^3 \\ 1 \\ \beta^4 \\ \beta^7 \\ \beta^{11} \\ \beta^3 \\ \beta^{10} \\ \beta \\ \beta^{13} \\ \beta^{10} \\ \beta \\ \beta^{13} \\ \beta^{13} \end{array}$	0 0 0 0 0 1 0 0 0 0 0 0 0	0 0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 1 0 0 0 0 0 0 0	$\begin{array}{c} \beta^3\\ \beta^9\\ \beta^6\\ \beta^3\\ \beta^9\\ \beta^6\\ \beta^5\\ 1\\ \beta^2\\ \beta^{13}\\ \beta^{10}\\ \beta^{14} \end{array}$	$\begin{array}{c} \beta^{12} \\ \beta^{3} \\ 1 \\ \beta^{12} \\ \beta^{3} \\ 1 \\ \beta^{12} \\ \beta^{4} \\ \beta^{14} \\ \beta^{9} \\ \beta^{6} \\ \beta^{10} \end{array}$	β^{10} β β^{13} β^{10} β β^{13} β^{7} β^{11} β^{3} β^{3} 1 β^{4}	0 0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0 0 0 0 0 1	$\begin{array}{c} \beta^3\\ \beta^9\\ \beta^6\\ \beta^3\\ \beta^9\\ \beta^6\\ \beta^{13}\\ \beta^{10}\\ \beta^{14}\\ \beta^5\\ 1\\ \beta^2\end{array}$	$egin{array}{c} & eta^{12} & \ & eta^{3} & \ & \ & \ & \ & \ & \ & \ & \ & \ & $	β^{10} β β^{13} β^{10} β β^{3} 1 β^{4} β^{7} β^{311} β^{3}	_
											[,	3	β^1	0	β^8		$\left[\begin{array}{c} \beta^2 \\ \beta^8 \\ \beta^5 \end{array} \right]$	=		$egin{smallmatrix} eta^3\ eta^9\ eta^6 \end{smallmatrix}$	$egin{array}{c} eta^{12} \ eta^{3} \ 1 \end{array}$	$\beta^2 \beta$	10 3 13	

Messages and codewords

(1	β	$, \beta^2$	$^2, \beta$	β^{12}, μ	$3^{14},$	$\beta^{12})$	()	3, 1	,0	$,\beta^9,$	β^{14} ,	$,\beta)$												
	1	0	0	β^5	eta^{12}	eta^7	0	0	0	β^{13}	eta^9	β^3	0	0	0	β^3	eta^{12}	eta^{10}	0	0	0	eta^3	eta^{12}	β^{10}
$(1,\beta,\beta^2)$	0	1	0	1	eta^4	eta^{11}	0	0	0	eta^{10}	eta^6	1	0	0	0	β^9	eta^{3}	eta	0	0	0	eta^{9}	eta^3	eta
	0	0	1	β^2	eta^{14}	eta^{3}	0	0	0	β^{14}	eta^{10}	eta^4	0	0	0	β^6	1	eta^{13}	0	0	0	eta^6	1	β^{13}
	0	0	0	β^{13}	β^9	β^3	1	0	0	β^5	eta^{12}	β^7	0	0	0	β^3	eta^{12}	eta^{10}	0	0	0	eta^3	eta^{12}	β^{10}
$(\beta, 1, 0)$	0	0	0	β^{10}	eta^6	1	0	1	0	1	eta^4	eta^{11}	0	0	0	β^9	eta^3	eta	0	0	0	eta^9	eta^3	β
	0	0	0	β^{14}	eta^{10}	eta^4	0	0	1	β^2	eta^{14}	eta^3	0	0	0	β^6	1	eta^{13}	0	0	0	eta^6	1	β^{13}
(- 0	0	0	0	β^3	eta^{12}	eta^{10}	0	0	0	β^3	eta^{12}	eta^{10}	1	0	0	β^5	eta^{12}	β^7	0	0	0	eta^{13}	eta^9	β^3
$(\beta^2, 0, \beta)$	0	0	0	β^9	eta^{3}	eta	0	0	0	β^9	eta^{3}	eta	0	1	0	1	eta^4	eta^{11}	0	0	0	eta^{10}	eta^6	1
	0	0	0	β^6	1	eta^{13}	0	0	0	β^6	1	eta^{13}	0	0	1	β^2	eta^{14}	eta^3	0	0	0	eta^{14}	eta^{10}	β^4
	0	0	0	β^3	eta^{12}	eta^{10}	0	0	0	β^3	eta^{12}	eta^{10}	0	0	0	β^{13}	β^9	eta^3	1	0	0	eta^5	β^{12}	β^7
$(0,\beta,1)$	0	0	0	β^9	eta^3	eta	0	0	0	β^9	eta^3	eta	0	0	0	β^{10}	β^6	1	0	1	0	1	β^4	β^{11}
	0	0	0	β^6	1	eta^{13}	0	0	0	β^6	1	eta^{13}	0	0	0	β^{14}	eta^{10}	eta^4	0	0	$1 \mid$	eta^2	eta^{14}	β^3
																	Г	β^2]		Г	β^3	β^{12}	2 B	10 J
												Г	C	01	0	08	, ′	08		'	09	ρ	P	
												ļ	3	β^{1}	0	β°			=	/	60	β°	,	5
																		$\beta^{\mathfrak{s}}$			β^6	1	β	13

Mide	dle	9-I	ay	ver	era	sur	e	CO	rr	ecti	ion													
(β	$, \beta^2$	2,	- F	,/	$\beta^{12})$	(β	$^{3}, 1$,0	$,\beta^9,$	β^{14}	$,\beta)$				- 0	- 10	- 10				- 9	- 10	-10 7
$(1,eta,eta^2)$	$\begin{array}{c}1\\0\\0\end{array}$	0 1 0	0 0 1	$\begin{array}{c c} \beta^5 \\ 1 \\ \beta^2 \\ \end{array}$	$\frac{\beta^{12}}{\beta^4}$ $\frac{\beta^{14}}{\beta^{14}}$	$\frac{\beta^7}{\beta^{11}}$ $\frac{\beta^3}{\beta^3}$	0 0 0	0 0 0	0 0 0	$\frac{\beta^{13}}{\beta^{10}}$ $\frac{\beta^{14}}{\beta^{5}}$	$egin{array}{c} eta^9 \ eta^6 \ eta^{10} \ \hline eta^{12} \end{array}$	$egin{array}{c} eta^3 \ 1 \ eta^4 \ \hline 0^7 \end{array}$	0 0 0	0 0 0	0 0 0	$egin{array}{c} eta^3 \ eta^9 \ eta^6 \ \hline eta^3 \ eta^6 \ \hline eta^3 \ eta^3 \ eta^6 \ \hline eta^3 \ eba^3 \ eba^3$	$\frac{\beta^{12}}{\beta^3}$ $\frac{1}{2}$	$egin{array}{c} eta^{10} & & \ eta^{13} & & \ eta^{10} & & \ eba^{10} & $	0 0 0	0 0 0	0 0 0	$\frac{\beta^3}{\beta^9}$ $\frac{\beta^6}{\beta^3}$	$\frac{\beta^{12}}{\beta^3}$ $\frac{1}{\beta^{12}}$	$egin{array}{c} eta^{10} \\ eta \\ eta^{13} \\ \hline eta^{10} \end{array}$
(eta,1,0)	$\begin{array}{c} 0\\ 0\\ 0\\ \hline 0 \end{array}$	0 0 0	0 0 0	$\begin{array}{c}\beta^{10}\\\beta^{10}\\\beta^{14}\\\hline\end{array}$	β^{6} β^{6} β^{10}	β^{0} 1 β^{4} 210	$ \begin{array}{c} 1\\ 0\\ 0\\ \end{array} $	0 1 0	0 0 1	β° 1 β^{2} β^{3}	$\frac{\beta^{12}}{\beta^4}$ $\frac{\beta^{14}}{\beta^{12}}$	β^{11} β^{3}	0 0 0	0 0 0	0 0 0	β^{9} β^{6}	β^{12} β^{3} $\frac{1}{2}$	β^{10} β β^{13}	0 0 0	0 0 0	0 0 0	β^{6} β^{9} β^{6}	β^{12} β^{3} 1 β^{9}	$\frac{\beta^{10}}{\beta}$ $\frac{\beta^{13}}{\beta^{3}}$
$(eta^2,0,eta)_{a}$	$\begin{array}{c} 0\\ 0\\ 0\\ \hline 0 \end{array}$	0 0 0	0 0 0	$\begin{array}{c}\beta^{9}\\\beta^{9}\\\beta^{6}\\\hline \alpha^{3}\end{array}$	β^{12} β^{3} 1 212	$\frac{\beta^{10}}{\beta}$ $\frac{\beta^{13}}{2^{10}}$	0 0 0	0 0 0	0 0 0	β^{9} β^{6}	β^{12} β^{3} $\frac{1}{2}$	β^{10} β β^{13}	$\begin{array}{c}1\\0\\0\end{array}$	0 1 0	0 0 1	β° 1 β^{2} β^{13}	β^{12} β^{4} β^{14}	β^{11} β^{3} β^{3}	0 0 0	0 0 0	0 0 0	$\frac{\beta^{10}}{\beta^{10}}$ $\frac{\beta^{10}}{\beta^{14}}$	$\frac{\beta^6}{\beta^{10}}$	$\frac{\beta^{3}}{1}$ $\frac{\beta^{4}}{27}$
(0,eta,1)	0 0 0	0 0 0	0 0 0	$egin{array}{c} eta^{9} \ eta^{6} \end{array}$	$egin{array}{c} eta^3 \ 1 \end{array}$	$egin{array}{c} eta^{=5} \ eta \ eta^{13} \end{array}$	0 0 0	0 0 0	0 0 0	$eta^9\ eta^6$	$egin{array}{c} eta^3 \ 1 \end{array}$	$egin{array}{c} eta^{13} \ eta^{13} \end{array}$	0 0 0	0 0 0	0 0 0	$egin{array}{c} eta^{10} \ eta^{14} \end{array}$	$egin{array}{c} eta^{6} \ eta^{10} \end{array}$	$egin{array}{c} eta^{3} \ eta^{4} \ eta^{4} \end{array}$	1 0 0	$\begin{array}{c} 0 \\ 1 \\ 0 \end{array}$	$\begin{array}{c} 0\\ 0\\ 1\end{array}$	$egin{array}{c} eta^2 \ eta^2 \end{array}$	$egin{array}{c} eta^{4} \ eta^{14} \end{array}$	$\begin{bmatrix} \beta^1 \\ \beta^{11} \\ \beta^3 \end{bmatrix}$
												[/	3	β^{1}	0	β^8		$\left[egin{smallmatrix} & eta^2 \ & eta^8 \ & eta^5 \end{bmatrix} ight]$	=	4 4 1	3 ³ 3 ⁹ 3 ⁶	$egin{array}{c} eta^{12} \ eta^{3} \ 1 \end{array} \ 1 \end{array}$	$\beta^2 \beta$	$\begin{bmatrix} 10 \\ \beta \\ 13 \end{bmatrix}$

Mide	dle	e-l	ay	ver	era	sur	e	CO	rr	ect	ion													
(β	$, \beta^{2}$	2,		, /	$\beta^{12})$	(£	3, 1	,0	$, \beta^9$	$, eta^{14}$	(β)												
$(1,eta,eta^2)$	1 0 0	0 1 0	0 0 1	$egin{array}{c} eta^5 \ 1 \ eta^2 \end{array}$	$egin{array}{c} eta^{12} \ eta^4 \ eta^{14} \end{array}$	$\beta^7\\\beta^{11}\\\beta^3$	0 0 0	0 0 0	0 0 0	$egin{array}{c c} eta^{13} \ eta^{10} \ eta^{14} \end{array}$	$egin{smallmatrix} eta^9\ eta^6\ eta^{10} \end{split}$	$egin{array}{c} eta^3 \ 1 \ eta^4 \end{array}$	0 0 0	0 0 0	0 0 0	$egin{smallmatrix} eta^3\ eta^9\ eta^6 \end{split}$	$egin{smallmatrix} eta^{12}\ eta^3\ 1 \end{split}$	$egin{array}{c} eta^{10} \ eta \ eta^{13} \ eta^{13} \end{array}$	0 0 0	0 0 0	0 0 0	$egin{smallmatrix} eta^3\ eta^9\ eta^6 \end{split}$	$egin{array}{c} eta^{12} \ eta^3 \ 1 \end{array}$	$\left. egin{smallmatrix} eta^{10} \ eta \ eta^{13} \end{bmatrix} ight.$
(eta,1,0)	0 0 0	0 0 0	0 0 0	$egin{array}{c} eta^{13} \ eta^{10} \ eta^{14} \end{array}$	$egin{array}{c} eta^9\ eta^6\ eta^{10} \end{array}$	$egin{array}{c} eta^3 \ 1 \ eta^4 \end{array}$	1 0 0	0 1 0	0 0 1	$egin{array}{c c} & eta^5 \ 1 \ & eta^2 \end{array}$	$egin{array}{c} eta^{12} \ eta^4 \ eta^{14} \end{array}$	$egin{array}{c} eta^7\ eta^{11}\ eta^3 \end{array}$	0 0 0	0 0 0	0 0 0	$egin{array}{c} eta^3\ eta^9\ eta^6 \end{array}$	$\beta^{12}\\\beta^{3}\\1$	$egin{array}{c} eta^{10} \ eta \ eta^{13} \ eta^{13} \end{array}$	0 0 0	0 0 0	0 0 0	$egin{smallmatrix} eta^3\ eta^9\ eta^6 \end{split}$	$\begin{array}{c}\beta^{12}\\\beta^{3}\\1\end{array}$	$egin{array}{c} eta^{10} \ eta \ eta^{13} \end{array} eta^{13}$
$(eta^2,0,eta)$	0 0 0	0 0 0	0 0 0	$egin{array}{c} eta^3\ eta^9\ eta^6 \end{array}$	$egin{array}{c} eta^{12} \ eta^3 \ 1 \end{array}$	$egin{array}{c} eta^{10} \ eta \ eta^{13} \end{array}$	0 0 0	0 0 0	0 0 0	$egin{array}{c} eta^3\ eta^9\ eta^6 \end{array}$	$egin{array}{c} eta^{12} \ eta^3 \ 1 \end{array}$	$egin{array}{c} eta^{10} \ eta \ eta^{13} \ eta^{13} \end{array}$	$\begin{array}{c}1\\0\\0\end{array}$	0 1 0	0 0 1	$egin{array}{c} eta^5 \ 1 \ eta^2 \end{array}$	$egin{array}{c} eta^{12} \ eta^4 \ eta^{14} \end{array}$	$egin{array}{c} eta^7 \ eta^{11} \ eta^3 \end{array}$	0 0 0	0 0 0	0 0 0	$\beta^{13}\\\beta^{10}\\\beta^{14}$	$egin{array}{c} eta^9 \ eta^6 \ eta^{10} \end{array}$	$\begin{array}{c} \overline{}^3 \\ 1 \\ \beta^4 \end{array}$
(0,eta,1)	0 0 0	0 0 0	0 0 0	$egin{array}{c} eta^3\ eta^9\ eta^6 \end{array}$	$\begin{array}{c}\beta^{12}\\\beta^{3}\\1\end{array}$	$egin{array}{c} eta^{10} \ eta \ eta^{13} \end{array}$	0 0 0	0 0 0	0 0 0	β^{6}	$(\beta_{\tilde{\mu}}^{11})$	$, \beta^7, \beta^7, \beta^{13}$	ອ້)ີງ 0	0 0 0	0 0 0	$\beta^{13}\\\beta^{10}\\\beta^{14}$	$egin{array}{c} eta^9 \ eta^6 \ eta^{10} \end{array}$	$egin{array}{c} eta^3 \ 1 \ eta^4 \end{array}$	1 0 0	0 1 0	0 0 1	$egin{array}{c} eta^5 \ 1 \ eta^2 \end{array}$	$egin{array}{c} eta^{12} \ eta^4 \ eta^{14} \end{array}$	$ \begin{bmatrix} \beta^7 \\ \beta^{11} \\ \beta^3 \end{bmatrix} $
												[/	3	β^1	0	β^8		$\left. egin{smallmatrix} \beta^2 \ \beta^8 \ \beta^5 \end{bmatrix} ight.$	_		eta^3 eta^9 eta^6	$egin{array}{c} eta^{12} \ eta^{3} \ 1 \end{array}$	$\beta^2 \beta$	$\begin{bmatrix} 10 \\ \beta \\ 13 \end{bmatrix}$
																			(0	G	ρ	2		<i>(</i> 12

 $(e_1, \beta, \beta^2, e_2, e_3, \beta^{12}) + (\beta^5, \beta^{14}, \beta^{12}) + (\beta^{11}, \beta^7, \beta) = (e'_1, \beta, \beta^2, e'_2, e'_3, \beta)$

Mid	dle	e-l	ay	er	era	sur	' e (CO	rr	ect	ion													
(β	$, \beta^{2}$	2,		,	β^{12})	(£	3, 1	,0	$, \beta^9$	β^{14}	$,\beta)$				0	10	10				0	10	10 -
$(1,eta,eta^2)$	$\begin{array}{c}1\\0\\0\end{array}$	0 1 0	0 0 1	$egin{array}{c} eta^3 \ 1 \ eta^2 \end{array}$	$egin{array}{c} eta^{12} \ eta^4 \ eta^{14} \end{array}$	$egin{array}{c} eta' \ eta^{11} \ eta^3 \end{array}$	0 0 0	0 0 0	0 0 0	$egin{array}{c} eta^{13} \ eta^{10} \ eta^{14} \end{array}$	$egin{array}{c} eta^9\ eta^6\ eta^{10} \end{array}$	$egin{array}{c} eta^3 \ 1 \ eta^4 \end{array}$	0 0 0	0 0 0	0 0 0	$egin{smallmatrix} eta^3\ eta^9\ eta^6 \end{split}$	$egin{array}{c} eta^{12} \ eta^3 \ 1 \end{array}$	$egin{array}{c} eta^{10} \ eta \ eta^{13} \ eta^{13} \end{array}$	0 0 0	0 0 0	0 0 0	$egin{smallmatrix} eta^3\ eta^9\ eta^6 \end{smallmatrix}$	$egin{array}{c} eta^{12} \ eta^3 \ 1 \end{array}$	$egin{array}{c} eta^{10} & \ eta & \ eta^{13} & \ eta^{13} & \ \end{array}$
(eta,1,0)	0 0 0	0 0 0	0 0 0	$egin{smallmatrix} eta^{13}\ eta^{10}\ eta^{14} \end{split}$	$egin{array}{c} eta^9\ eta^6\ eta^{10} \end{array}$	$egin{array}{c} eta^{3} \ 1 \ eta^{4} \end{array}$	$\begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$	0 1 0	0 0 1	$egin{array}{c} eta^5 \ 1 \ eta^2 \end{array}$	$egin{array}{c} eta^{12} \ eta^4 \ eta^{14} \end{array}$	$egin{array}{c} eta^7 \ eta^{11} \ eta^3 \end{array}$	0 0 0	0 0 0	0 0 0	$egin{array}{c} eta^3\ eta^9\ eta^6 \end{array}$	$\begin{array}{c}\beta^{12}\\\beta^{3}\\1\end{array}$	$egin{array}{c} eta^{10} \ eta \ eta^{13} \end{array}$	0 0 0	0 0 0	0 0 0	$egin{array}{c} eta^3\ eta^9\ eta^6 \end{array}$	$egin{array}{c} eta^{12} \ eta^3 \ 1 \end{array}$	$egin{array}{c} eta^{10} \ eta \ eta^{13} \end{array} eta^{13}$
$(eta^2,0,eta)$	$ \begin{array}{c c} 0\\ 0\\ 0 \end{array} $	0 0 0	0 0 0	$egin{array}{c} eta^{3} \ eta^{9} \ eta^{6} \end{array}$	$egin{array}{c} eta^{12} \ eta^3 \ 1 \end{array}$	$egin{array}{c} eta^{10} \ eta \ eta^{13} \end{array}$	0 0 0	0 0 0	0 0 0	$egin{array}{c} eta^3\ eta^9\ eta^6 \end{array}$	$egin{array}{c} eta^{12} \ eta^3 \ 1 \end{array}$	$egin{array}{c} eta^{10} \ eta \ eta^{13} \ eta^{13} \end{array}$	1 0 0	0 1 0	0 0 1	$egin{array}{c} eta^5 \ 1 \ eta^2 \end{array}$	$egin{array}{c} eta^{12} \ eta^4 \ eta^{14} \end{array}$	$egin{array}{c} eta^7 \ eta^{11} \ eta^3 \end{array}$	0 0 0	0 0 0	0 0 0	$egin{array}{c} eta^{13} \ eta^{10} \ eta^{14} \end{array}$	$egin{array}{c} eta^9 \ eta^6 \ eta^{10} \end{array}$	$\begin{array}{c} \beta^3 \\ 1 \\ \beta^4 \end{array}$
(0,eta,1)	0 0 0	0 0 0	0 0 0	$egin{array}{c} eta^3 \ eta^9 \ eta^6 \end{array}$	$\beta^{12} \\ \beta^{3} \\ 1$	$egin{array}{c} eta^{10} \ eta^{10} \ eta^{13} \ eta^{13} \end{array}$	0 0 0	0 0 0	0 0 0	$egin{array}{c} eta^3 \ eta^9 \ eta^6 \end{array}$	$\beta^{12} \\ \beta^{3} \\ 1$	$egin{array}{c} eta^{10} \ eta \ eta^{13} \ eta^{13} \end{array}$	0 0 0	0 0 0	0 0 0	$egin{array}{c} eta^{13} \ eta^{10} \ eta^{14} \end{array}$	$egin{array}{c} eta^9 \ eta^6 \ eta^{10} \end{array}$	$egin{array}{c} eta^3 \ 1 \ eta^4 \end{array}$	1 0 0	0 1 0	0 0 1	$egin{array}{c} eta^5 \ 1 \ eta^2 \end{array}$	$egin{array}{c} eta^{12} \ eta^4 \ eta^{14} \end{array}$	$\begin{bmatrix} \beta^7 \\ \beta^{11} \\ \beta^3 \end{bmatrix}$
												[/	3	β^1	0	β^8]	$egin{smallmatrix} eta^2 \ eta^8 \ eta^5 \end{bmatrix}$	=		$egin{smallmatrix} & eta^3 \ & eta^9 \ & eta^6 \end{smallmatrix}$	$egin{array}{c} eta^{12} \ eta^{3} \ 1 \end{array}$	$\beta^2 \beta$	$\begin{bmatrix} 10 \\ \beta \\ 13 \end{bmatrix}$
																			$(e_1$	ι,β	β, β^2 +	$^{2}, e_{2}, \\ \beta^{5}, \beta^{5},$	$\beta^{14}, \beta^{14}, \beta^{14}$	β^{12}
																			= ($[e_1']$	$^+, \beta,$	$- \left(\beta^1 \\ \beta^2, \epsilon \right)$	$egin{array}{c} {}^1, eta^{7}, {eta^{7}}, e^{\prime}_3, e^{\prime}_3 \end{array}$,eta),eta)



Mid	dle	-lay	/er	par	ity-	ch	eck	eq	uat	ion	S										
(1	$1, \beta, \beta$	β^2, μ	β^{12}, μ	$3^{14},$	$\beta^{12})$	$(\beta,$	1, 0	$, \beta^9$	β^{14}	$,\beta)$									_		
$(1,eta,eta^2)$	$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$) 0 1 0) 1	$\begin{vmatrix} \beta^5 \\ 1 \\ \beta^2 \end{vmatrix}$	$egin{array}{c} eta^{12} \ eta^4 \ eta^{14} \end{array}$	$egin{array}{c} eta^7 \ eta^{11} \ eta^3 \end{array} \end{array}$	0 (0 (0 () () () () () ()	$\begin{vmatrix} \beta^{13} \\ \beta^{10} \\ \beta^{14} \end{vmatrix}$	$egin{array}{c} eta^9\ eta^6\ eta^{10} \end{array}$	$egin{array}{c} eta^3 \ 1 \ eta^4 \end{array}$	0 0 0	0 0 0 0 0 0	$egin{array}{c c} & \beta^3 \ & \beta^9 \ & \beta^6 \end{array}$	$\begin{array}{c}\beta^{12}\\\beta^{3}\\1\end{array}$	$egin{array}{c} eta^{10} \ eta \ eta^{13} \ eta^{13} \end{array}$	0 0 0	0 0 0	0 0 0	$egin{smallmatrix} eta^3\ eta^9\ eta^6 \end{smallmatrix}$	$egin{array}{c} eta^{12} \ eta^3 \ 1 \end{array}$	$\left.\begin{array}{c}\beta^{10}\\\beta\\\beta^{13}\end{array}\right $
(eta,1,0)	0 (0 (0 (0 0 0 0	$egin{array}{c} eta^{13} \ eta^{10} \ eta^{14} \end{array}$	$egin{array}{c} eta^9 \ eta^6 \ eta^{10} \end{array}$	$egin{array}{c} eta^3 \ 1 \ eta^4 \end{array}$	1 (0 1 0 () 0 L 0) 1	$\begin{array}{c c} \beta^5 \\ 1 \\ \beta^2 \end{array}$	$egin{array}{c} eta^{12} \ eta^4 \ eta^{14} \end{array}$	$egin{array}{c} eta^7 \ eta^{11} \ eta^3 \end{array}$	0 0 0	0 0 0 0 0 0	$egin{array}{c c} & \beta^3 \ & \beta^9 \ & \beta^6 \end{array}$	$\begin{array}{c}\beta^{12}\\\beta^{3}\\1\end{array}$	$egin{array}{c} eta^{10} \ eta \ eta^{13} \ eta^{13} \end{array}$	0 0 0	0 0 0	0 0 0	$egin{array}{c} eta^3 \ eta^9 \ eta^6 \end{array}$	$egin{array}{c} eta^{12} \ eta^3 \ 1 \end{array}$	$\begin{array}{c} \overline{\beta^{10}} \\ \beta \\ \beta^{13} \end{array}$
$(eta^2,0,eta)$	0 (0 (0 (0 0 0 0	$egin{array}{c} eta^3 \ eta^9 \ eta^6 \end{array}$	$egin{array}{c} eta^{12} \ eta^3 \ 1 \end{array}$	$egin{array}{c} eta^{10} \ eta \ eta^{13} \end{array}$	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0) 0) 0) 0	$egin{array}{c} eta^3 \ eta^9 \ eta^6 \end{array}$	$\begin{array}{c}\beta^{12}\\\beta^{3}\\1\end{array}$	$egin{array}{c} eta^{10} \ eta \ eta^{13} \ eta^{13} \end{array}$	1 0 0	$ \begin{array}{ccc} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{array} $	$egin{array}{c c} & \beta^5 \ 1 \ & \beta^2 \end{array}$	$egin{array}{c} eta^{12} \ eta^4 \ eta^{14} \end{array}$	$egin{array}{c} eta^7 \ eta^{11} \ eta^3 \end{array}$	0 0 0	0 0 0	0 0 0	$egin{array}{c} eta^{13} \ eta^{10} \ eta^{14} \end{array}$	$egin{array}{c} eta^9 \ eta^6 \ eta^{10} \end{array}$	$\frac{\beta^3}{1}\\\beta^4$
(0,eta,1)	0 (0 (0 () ()) ()) ()	$egin{array}{c} eta^3 \ eta^9 \ eta^6 \end{array}$	$\begin{array}{c}\beta^{12}\\\beta^{3}\\1\end{array}$	$egin{array}{c} eta^{10} \ eta \ eta^{13} \ eta^{13} \end{array}$	0 (0 (0 () 0) 0) 0	$egin{array}{c} eta^3 \ eta^9 \ eta^6 \end{array}$	$\begin{array}{c}\beta^{12}\\\beta^{3}\\1\end{array}$	$egin{array}{c} eta^{10} & \ eta^{13} & \ eba^{13} & $	0 0 0	0 0 0 0 0 0	$egin{array}{c} eta^{13} \ eta^{10} \ eta^{14} \end{array}$	$egin{array}{c} eta^9 \ eta^6 \ eta^{10} \end{array}$	$egin{array}{c} eta^3 \ 1 \ eta^4 \end{array}$	$\begin{array}{c}1\\0\\0\end{array}$	0 1 0	0 0 1	$egin{array}{c} eta^5 \ 1 \ eta^2 \end{array}$	$egin{array}{c} eta^{12} \ eta^4 \ eta^{14} \end{array}$	$ \begin{array}{c} \beta^7 \\ \beta^{11} \\ \beta^3 \end{array} \right] $
	,		$\begin{bmatrix} \beta^{\xi} \\ 1 \\ \beta^{\xi} \end{bmatrix}$	$\beta \beta $	12 3^4 14	$eta^7_{eta^{11}}_{eta^3}$	$egin{array}{c} eta^9 \ eta^6 \ eta^{10} \end{array}$			[/	3	β^{10}	β^8] [$\left. egin{smallmatrix} eta^2 \ eta^8 \ eta^5 \end{bmatrix} ight.$	=	یم کر کر	33 39 36	$\begin{array}{c} \beta^{12} \\ \beta^{3} \\ 1 \end{array}$	eta^{2}_{eta}	10 } 13
$(e_1^{\prime},\beta,\beta^2,$	e_2', e_2'	β_3, β_1	$) \begin{bmatrix} \rho \\ 1 \\ 0 \\ 0 \end{bmatrix}$) 1)	$egin{array}{c} 0 \\ 0 \\ 1 \end{array}$	$egin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}$		_							$(e_1$,β,	$, \beta^2 + ($	$,e_2,\ eta^5,eta$	$e_3, ar{\epsilon}_3, ar{\epsilon}_3^{14}, ar{\epsilon}_3^{14}$	$\beta^{12})$
(4	e_1^\prime, e_2^\prime	e_3, e_3') =	(1,	3 ¹⁰ ,	$\beta^7)$:	= ($e_1',$	$+$ β, β	$egin{array}{c} (eta^{11}\ eta^2,e_2^\prime) \ eta^2,e_2^\prime \end{array}$	$,eta^7,eta^7, e_3'$,eta),eta)

Heterogeneity

Parameters

$$\begin{split} & (n_{1,1}, n_{1,2}, n_{2,1}, n_{2,2}) = (10,11,10,10) \\ & (n_{3,1}, n_{3,2}, n_{3,3}, n_{3,4}) = (12,12,12,12) \\ & (k_{1,1}, k_{1,2}, k_{2,1}, k_{2,2}) = (6,6,7,7) \\ & (k_{3,1}, k_{3,2}, k_{3,3}, k_{3,4}) = (9,8,9,9) \\ & (r_{1,1}, r_{1,2}, r_{2,1}, r_{2,2}) = (4,5,3,3) \\ & (r_{3,1}, r_{3,2}, r_{3,3}, r_{3,4}) = (3,4,3,3) \\ & (\delta_{1,1}, \delta_{1,2}) = (\delta_{2,1}, \delta_{2,2}) = (1,1) \\ & \delta_1 = \delta_2 = \delta_{1,1} + \delta_{1,2} = 2 \\ & (\delta_{3,1}, \delta_{3,2}, \delta_{3,3}, \delta_{3,4}) = (1,2,1,1) \\ & \delta_3 = 1 + 2 + 1 + 1 = 5 \\ & (\gamma_1, \gamma_2, \gamma_3) = (1, \frac{1}{2}, \frac{1}{2}) \\ & \gamma = 2 \cdot (1) + 2 \cdot (\frac{1}{2}) + 4 \cdot (\frac{1}{2}) = 5 \end{split}$$

Unequal error protection

- 1st layer error correction capability > $d_{1,x,i} = r_{x,i} - \delta_{x,i} - 2\gamma_x + 1$
- 2nd layer error correction capability > $d_{2,x,i} = r_{x,i} - \delta_{x,i} + \delta_x + 1$ > $\delta_x = \delta_{x,1} + \dots + \delta_{x,p_x}$
- **3**rd layer error correction capability > $d_{3,x,i} = r_{x,i} - \delta_{x,i} + \delta_x - p_x \gamma_x + \gamma + 1$ > $\gamma = p_1 \gamma_1 + \dots + p_{p_0} \gamma_{p_0}$

[$\overline{2}$	3	$\mid 2$	2	2	2	2	2
	6	7	5	5	8	8	8	8
	9	10	9	9	11	11	11	11



Add a cloud to the existing network



- Add a cloud to the existing network
 - Step 1: parameter selection
 - Cloud 4 chooses its local parameters

$$\mathbf{T}_4 = \begin{bmatrix} \mathbf{A}_{4,4} & \mathbf{B}_{4,1} & \mathbf{B}_{4,2} & \mathbf{B}_{4,3} \\ \hline \mathbf{U}_4 & & \end{bmatrix}$$



Add a cloud to the existing network

- Step 1: parameter selection
 - Cloud 4 chooses its local parameters
- Step 2: information exchange

• Uplink




Scalability

Add a cloud to the existing network

- Step 1: parameter selection
 - Cloud 4 chooses its local parameters
- Step 2: information exchange
 - Uplink
 - Downlink

$$\mathbf{T}_4 = \begin{bmatrix} \mathbf{A}_{4,4} & \mathbf{B}_{4,1} & \mathbf{B}_{4,2} & \mathbf{B}_{4,3} \\ \hline \mathbf{U}_4 & & \end{bmatrix}$$



Scalability

Add a cloud to the existing network

- Step 1: parameter selection
 - Cloud 4 chooses its local parameters
- Step 2: information exchange
 - Uplink
 - Downlink
- Step 3: update
 - Local cloud *i* adds m₄B_{4,i}U_i
 to its original parity
 - Local cloud 4 computes its parity $\mathbf{m}_4 \mathbf{A}_{4,4} + \sum_{i=1}^3 \mathbf{m}_i \mathbf{B}_{i,4} \mathbf{U}_4$

$$\mathbf{T}_4 = \begin{bmatrix} \mathbf{A}_{4,4} & \mathbf{B}_{4,1} & \mathbf{B}_{4,2} & \mathbf{B}_{4,3} \\ \hline \mathbf{U}_4 & & \end{bmatrix}$$



Scalability

Add a cloud to the existing network

- Step 1: parameter selection
 - Cloud 4 chooses its local parameters
- Step 2: information exchange
 - Uplink
 - Downlink
- Step 3: update
 - Local cloud *i* adds m₄B_{4,i}U_i
 to its original parity
 - Local cloud 4 computes its parity $\mathbf{m}_4 \mathbf{A}_{4,4} + \sum_{i=1}^3 \mathbf{m}_i \mathbf{B}_{i,4} \mathbf{U}_4$
- Local parameters of other clouds remain unchanged

$$\mathbf{T}_4 = \begin{bmatrix} \mathbf{A}_{4,4} & \mathbf{B}_{4,1} & \mathbf{B}_{4,2} & \mathbf{B}_{4,3} \\ \hline \mathbf{U}_4 & & \end{bmatrix}$$











- The data of a local cloud become hot
 - Split the cloud into two smaller clouds



$$\mathbf{T}_x = egin{bmatrix} \mathbf{A}_{x,x} & \mathbf{B}_{x,1} & \dots & \mathbf{B}_{x,p} \ \hline \mathbf{U}_x & \mathbf{Z}_x \end{bmatrix}$$

- The data of a local cloud become hot
 - Split the cloud into two smaller clouds
 - Divide matrices G and T_{χ} into blocks





- The data of a local cloud become hot
 - Split the cloud into two smaller clouds
 - Divide matrices ${f G}$ and ${f T}_\chi$ into blocks
 - Reorder the blocks



- The data of a local cloud become hot
 - Split the cloud into two smaller clouds
 - Divide matrices **G** and \mathbf{T}_{χ} into blocks
 - Reorder the blocks and obtain new matrices $\mathbf{T}_{\chi^{a}}$, $\mathbf{T}_{\chi^{b}}$, **G**





The data of a local cloud become hot

- Split the cloud into two smaller clouds
 - Divide matrices **G** and **T**_{*x*} into blocks
 - Reorder the blocks and obtain new matrices $\mathbf{T}_{x^{\mathrm{a}}}$, $\mathbf{T}_{x^{\mathrm{b}}}$, **G**
 - Other clouds remain unchanged
- Erasure correction capability
 - Old clouds: remains unchanged



$$\begin{split} \mathbf{A}_{1^{\mathrm{a}},1^{\mathrm{a}}} &= \mathbf{A}_{1,1} \left[1:k_{1}^{\mathrm{a}},1:r_{1}^{\mathrm{a}} \right], \\ \mathbf{B}_{1^{\mathrm{b}},1^{\mathrm{a}}} &= \mathbf{A}_{1,1} \left[k_{1}^{\mathrm{a}}+1:k_{1},1:\delta_{1}^{\mathrm{a}} \right], \\ \mathbf{A}_{1^{\mathrm{b}},1^{\mathrm{b}}} &= \mathbf{A}_{1,1} \left[k_{1}^{\mathrm{a}}+1:k_{1},,r_{1}^{\mathrm{a}}+1:r_{1} \right], \\ \mathbf{B}_{1^{\mathrm{a}},1^{\mathrm{b}}} &= \mathbf{A}_{1,1} \left[1:k_{1}^{\mathrm{a}},r_{1}^{\mathrm{a}}+1:r_{1}^{\mathrm{a}}+\delta_{1}^{\mathrm{b}} \right], \\ \mathbf{U}_{1}^{\mathrm{a}} &= \mathbf{U}_{1} \left[1:\delta_{1}^{\mathrm{a}},1:r_{1}^{\mathrm{a}} \right], \\ \mathbf{U}_{1}^{\mathrm{b}} &= \mathbf{U}_{1} \left[\delta_{1}^{\mathrm{a}}+1:\delta_{1},r_{1}^{\mathrm{a}}+1:r_{1}^{\mathrm{b}} \right], \end{split}$$



The data of a local cloud become hot

- Split the cloud into two smaller clouds
 - Divide matrices **G** and **T**_{*x*} into blocks
 - Reorder the blocks and obtain new matrices $\mathbf{T}_{x^{a}}$, $\mathbf{T}_{x^{b}}$, **G**
 - Other clouds remain unchanged
- Erasure correction capability
 - Old clouds: remains unchanged

• New clouds:
$$r_{\chi^{\mathrm{a}}} - \delta_{\chi^{\mathrm{a}}}$$
 , $r_{\chi^{\mathrm{b}}} - \delta_{\chi^{\mathrm{b}}}$ ($r_{\chi} - \delta_{\chi}$ in total)

$$\begin{split} \mathbf{A}_{1^{\mathrm{a}},1^{\mathrm{a}}} &= \mathbf{A}_{1,1} \left[1:k_{1}^{\mathrm{a}},1:r_{1}^{\mathrm{a}} \right], \\ \mathbf{B}_{1^{\mathrm{b}},1^{\mathrm{a}}} &= \mathbf{A}_{1,1} \left[k_{1}^{\mathrm{a}}+1:k_{1},1:\delta_{1}^{\mathrm{a}} \right], \\ \mathbf{A}_{1^{\mathrm{b}},1^{\mathrm{b}}} &= \mathbf{A}_{1,1} \left[k_{1}^{\mathrm{a}}+1:k_{1},,r_{1}^{\mathrm{a}}+1:r_{1} \right], \\ \mathbf{B}_{1^{\mathrm{a}},1^{\mathrm{b}}} &= \mathbf{A}_{1,1} \left[1:k_{1}^{\mathrm{a}},r_{1}^{\mathrm{a}}+1:r_{1}^{\mathrm{a}}+\delta_{1}^{\mathrm{b}} \right], \\ \mathbf{U}_{1}^{\mathrm{a}} &= \mathbf{U}_{1} \left[1:\delta_{1}^{\mathrm{a}},1:r_{1}^{\mathrm{a}} \right], \\ \mathbf{U}_{1}^{\mathrm{b}} &= \mathbf{U}_{1} \left[\delta_{1}^{\mathrm{a}}+1:\delta_{1},r_{1}^{\mathrm{a}}+1:r_{1}^{\mathrm{b}} \right], \end{split}$$





Outline

- Introduction
 - Latency-reliability trade-off in storage systems
 - Heterogeneity, scalability, and flexibility
- Preliminaries
 - Existing literature
 - Cauchy Reed-Solomon (CRS) codes
- Constructions
 - Double-level codes
 - Hierarchical codes
 - Properties

Conclusion

Conclusion

Main contribution

- Proposed CRS-based codes with hierarchical locality
- Showed that our construction achieves scalability, heterogeneity and flexibility, which are critical for practical cloud storage
- Proved that our construction requires a field size linear to the maximum codeword length

Future work

Extend erasure-correction to error-correction, which is useful in novel SSD solutions for multi-task-oriented applications, such as autonomous driving, where latency and reliability are both important

Thank you!