GLOBECOM 2019

Hierarchical Coding to Enable Scalability and Flexibility in Heterogeneous Cloud Storage

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Outline

- **Introduction**
  - Latency-reliability trade-off in storage systems
  - Heterogeneity, scalability, and flexibility
- **Preliminaries**
  - Existing literature
  - Cauchy Reed-Solomon (CRS) codes
- **Constructions**
  - Double-level codes
  - Hierarchical codes
  - Properties
- **Conclusion**

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  - IoT devices
  - In-memory analytics
  - Content delivery network

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  - RAID, flash memory
  - Cloud storage
  - Persistent memory, computational storage

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- **Distributed storage (we focus here on cloud storage)**
  - Low latency

Latency-Reliability Trade-off in Storage Systems

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- **Codes with hierarchical locality provides a trade-off between high reliability and low latency**

Practical Concerns of Modern Cloud Storage

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- **Flexibility**
  - The usage rate of a piece of data is not likely to remain unchanged in dynamic cloud storage
    - Cold data become hot, hot data become cold

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- **Scalability**
  - Enables adding a new local cloud without changing the encoding-decoding components (in the generator matrix) of the already-existing local clouds.
- **Flexibility**
  - Enables dynamic split of a local cloud into smaller clouds without worsening the global ECC capability nor changing the remaining components

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- **Generalized integrated interleaved (GII) Codes** [1][2]
  - Support a large set of error patterns
  - Distribution of the data symbols is highly restricted
  - Local codewords are equally protected

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- **Sum-rank codes** [4]
  - Maximal recoverability and flexibility
  - Field size exponential in the maximum codeword length

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Cauchy Reed-Solomon (CRS) Codes

- **CRS-based parity-check matrix with minimum distance (t+1)**

$$\mathbf{M} = \begin{bmatrix} \frac{1}{a_1-b_1} & \frac{1}{a_1-b_2} & \cdots & \frac{1}{a_1-b_r} & \cdots & \frac{1}{a_1-b_t} \\ \frac{1}{a_2-b_1} & \frac{1}{a_2-b_2} & \cdots & \frac{1}{a_2-b_r} & \cdots & \frac{1}{a_2-b_t} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{1}{a_s-b_1} & \frac{1}{a_s-b_2} & \cdots & \frac{1}{a_s-b_r} & \cdots & \frac{1}{a_s-b_t} \\ -1 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & -1 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -1 & \ldots & 0 \end{bmatrix}^{\mathrm{T}}.$$

Cauchy Reed-Solomon (CRS) Codes

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Cauchy matrix

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  - Cauchy matrix of size $s \times t$

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Negative identity matrix

- Cauchy matrix of size $s \times t$
- Negative identity matrix of size $r \times r$

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\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{1}{a_s-b_1} & \frac{1}{a_s-b_2} & \cdots & \frac{1}{a_s-b_r} & \cdots & \frac{1}{a_s-b_t} \\
-1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & -1 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -1 & \cdots & 0
\end{bmatrix}^{\mathrm{T}} .
$$

Zero matrix

- Cauchy matrix of size $s \times t$
- Negative identity matrix of size $r \times r$
- Zero matrix of size $r \times (t-r)$

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Double-Level Codes

- **Construction of double-level accessible codes based on CRS codes**

$$\mathbf{G} = \left[\begin{array}{c|c|c|c|c|c|c} \mathbf{I}_{k_1} & \mathbf{A}_{1,1} & \mathbf{0} & \mathbf{A}_{1,2} & \dots & \mathbf{0} & \mathbf{A}_{1,p} \\ \hline \mathbf{0} & \mathbf{A}_{2,1} & \mathbf{I}_{k_2} & \mathbf{A}_{2,2} & \dots & \mathbf{0} & \mathbf{A}_{2,p} \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \mathbf{0} & \mathbf{A}_{p,1} & \mathbf{0} & \mathbf{A}_{p,2} & \dots & \mathbf{I}_{k_p} & \mathbf{A}_{p,p} \end{array}\right].$$

$$\mathbf{T}_x = \left[\begin{array}{c|c|c|c} \mathbf{A}_{x,x} & \mathbf{B}_{x,1} & \dots & \mathbf{B}_{x,p} \\ \hline \mathbf{U}_x & \multicolumn{3}{c}{\mathbf{Z}_x} \end{array}\right]$$

Auxiliary Cauchy matrix

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  - $\mathbf{I}_{k_x}: k_x \times k_x$

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Identity matrices: generate systematic symbols

Construction of double-level accessible codes based on CRS codes
$\mathbf{I}_{k_x}: k_x \times k_x$

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  - $\mathbf{I}_{k_x}: k_x \times k_x$
  - $\mathbf{A}_{x,x}: k_x \times r_x$

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Generate local parities

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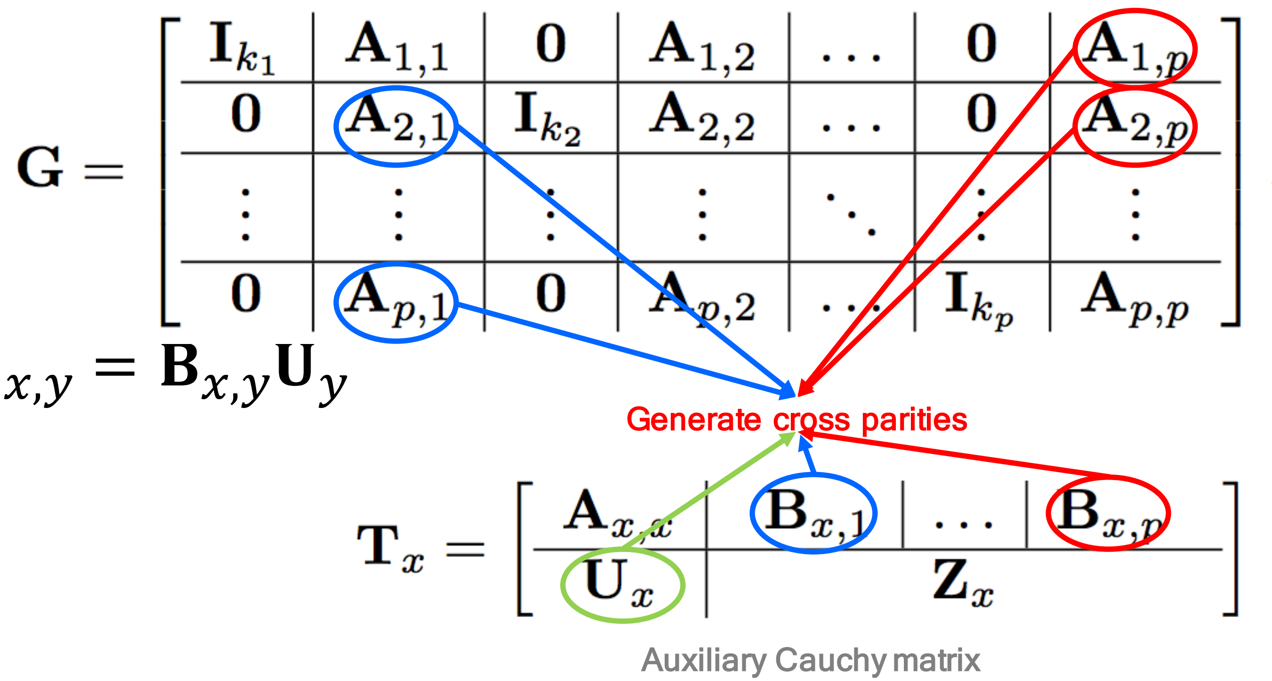

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- **Erasure correction capability**
  - 1st layer error correction capability
    - $d_{1,x} = r_x - \delta_x + 1$
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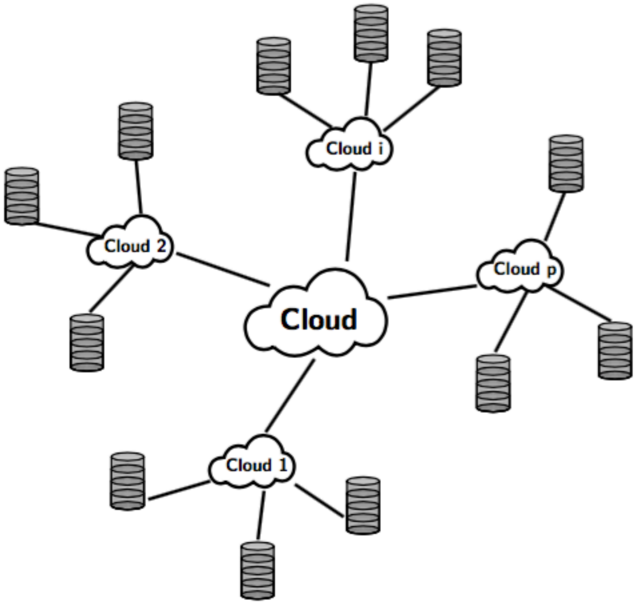

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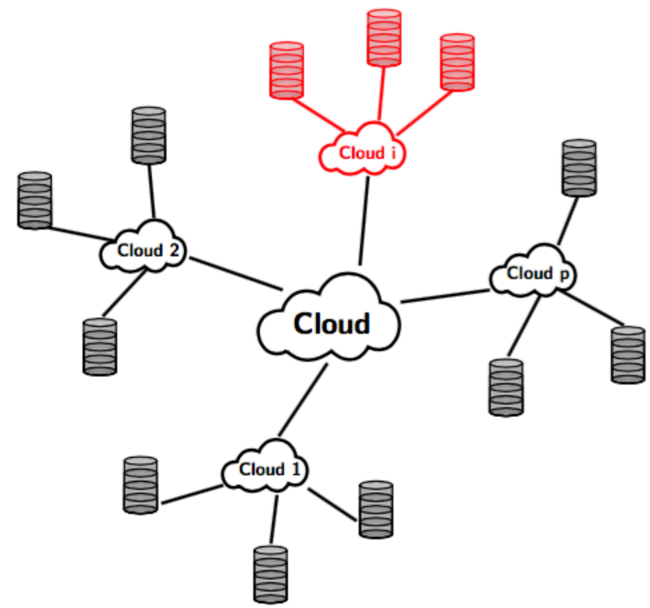

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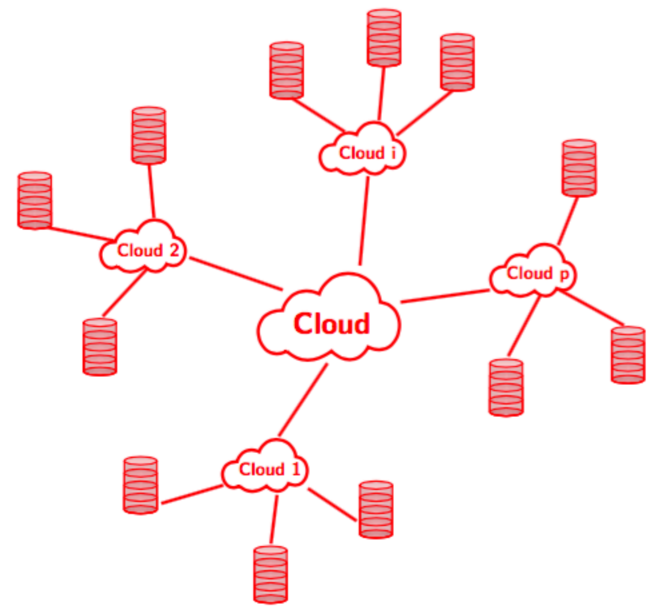


Example

- **Double-level codes**

$$
\left[\begin{array}{ccc|ccc|ccc|ccc}
1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 \\
0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 \\
0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 \\
\hline
0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 \\
0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} \\
0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3
\end{array}\right]
$$

$$
\mathbf{T}_1 = \mathbf{T}_2 = \left[\begin{array}{c|c} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \hline \mathbf{U}_1 & \mathbf{Z}_1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \hline \mathbf{U}_2 & \mathbf{Z}_2 \end{array}\right] = \left[\begin{array}{ccc|c}
\frac{1}{\beta-\beta^8} & \frac{1}{\beta-\beta^9} & \frac{1}{\beta-\beta^{10}} & \frac{1}{\beta-\beta^{11}} \\
\frac{1}{\beta^2-\beta^8} & \frac{1}{\beta^2-\beta^9} & \frac{1}{\beta^2-\beta^{10}} & \frac{1}{\beta^2-\beta^{11}} \\
\frac{1}{\beta^3-\beta^8} & \frac{1}{\beta^3-\beta^9} & \frac{1}{\beta^3-\beta^{10}} & \frac{1}{\beta^3-\beta^{11}} \\
\hline
\frac{1}{\beta^7-\beta^8} & \frac{1}{\beta^7-\beta^9} & \frac{1}{\beta^7-\beta^{10}} & \frac{1}{\beta^7-\beta^{11}}
\end{array}\right] = \left[\begin{array}{ccc|c}
\beta^5 & \beta^{12} & \beta^7 & \beta^9 \\
1 & \beta^4 & \beta^{11} & \beta^6 \\
\beta^2 & \beta^{14} & \beta^3 & \beta^{10} \\
\hline
\beta^4 & 1 & \beta^9 & \beta^7
\end{array}\right]
$$

Example

- **Local parities**

$$
\left[\begin{array}{ccc|ccc|ccc|ccc}
1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 \\
0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 \\
0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 \\
\hline
0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 \\
0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} \\
0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3
\end{array}\right]
$$

$$
\mathbf{T}_1 = \mathbf{T}_2 = \left[\begin{array}{c|c} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \hline \mathbf{U}_1 & \mathbf{Z}_1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \hline \mathbf{U}_2 & \mathbf{Z}_2 \end{array}\right] = \left[\begin{array}{ccc|c}
\frac{1}{\beta-\beta^8} & \frac{1}{\beta-\beta^9} & \frac{1}{\beta-\beta^{10}} & \frac{1}{\beta-\beta^{11}} \\
\frac{1}{\beta^2-\beta^8} & \frac{1}{\beta^2-\beta^9} & \frac{1}{\beta^2-\beta^{10}} & \frac{1}{\beta^2-\beta^{11}} \\
\frac{1}{\beta^3-\beta^8} & \frac{1}{\beta^3-\beta^9} & \frac{1}{\beta^3-\beta^{10}} & \frac{1}{\beta^3-\beta^{11}} \\
\hline
\frac{1}{\beta^7-\beta^8} & \frac{1}{\beta^7-\beta^9} & \frac{1}{\beta^7-\beta^{10}} & \frac{1}{\beta^7-\beta^{11}}
\end{array}\right] = \left[\begin{array}{ccc|c}
\beta^5 & \beta^{12} & \beta^7 & \beta^9 \\
1 & \beta^4 & \beta^{11} & \beta^6 \\
\beta^2 & \beta^{14} & \beta^3 & \beta^{10} \\
\hline
\beta^4 & 1 & \beta^9 & \beta^7
\end{array}\right]
$$

Example

- **Cross parities**

$$
\left[\begin{array}{ccc|ccc|ccc|ccc}
1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 \\
0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 \\
0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 \\
\hline
0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 \\
0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} \\
0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3
\end{array}\right]
$$

$$
\mathbf{T}_1 = \mathbf{T}_2 = \left[\begin{array}{c|c} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \hline \mathbf{U}_1 & \mathbf{Z}_1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \hline \mathbf{U}_2 & \mathbf{Z}_2 \end{array}\right] = \left[\begin{array}{ccc|c}
\frac{1}{\beta-\beta^8} & \frac{1}{\beta-\beta^9} & \frac{1}{\beta-\beta^{10}} & \frac{1}{\beta-\beta^{11}} \\
\frac{1}{\beta^2-\beta^8} & \frac{1}{\beta^2-\beta^9} & \frac{1}{\beta^2-\beta^{10}} & \frac{1}{\beta^2-\beta^{11}} \\
\frac{1}{\beta^3-\beta^8} & \frac{1}{\beta^3-\beta^9} & \frac{1}{\beta^3-\beta^{10}} & \frac{1}{\beta^3-\beta^{11}} \\
\hline
\frac{1}{\beta^7-\beta^8} & \frac{1}{\beta^7-\beta^9} & \frac{1}{\beta^7-\beta^{10}} & \frac{1}{\beta^7-\beta^{11}}
\end{array}\right] = \left[\begin{array}{ccc|c}
\beta^5 & \beta^{12} & \beta^7 & \beta^9 \\
1 & \beta^4 & \beta^{11} & \beta^6 \\
\beta^2 & \beta^{14} & \beta^3 & \beta^{10} \\
\hline
\beta^4 & 1 & \beta^9 & \beta^7
\end{array}\right]
$$

Example

- **Messages and codewords**

$$(1, \beta, \beta^2, \quad \beta^{14}, 0, 0) \quad (\beta, 1, 0, \quad \beta^6, 0, \beta^{13})$$

$$\begin{matrix} (1, \beta, \beta^2) \\ \\ (\beta, 1, 0) \end{matrix} \left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 \\ 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 \\ 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 \\ \hline 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 \\ 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} \\ 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 \end{array}\right]$$

$$\mathbf{T}_1 = \mathbf{T}_2 = \left[\begin{array}{c|c} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \hline \mathbf{U}_1 & \mathbf{Z}_1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \hline \mathbf{U}_2 & \mathbf{Z}_2 \end{array}\right] = \left[\begin{array}{ccc|c} \frac{1}{\beta-\beta^8} & \frac{1}{\beta-\beta^9} & \frac{1}{\beta-\beta^{10}} & \frac{1}{\beta-\beta^{11}} \\ \frac{1}{\beta^2-\beta^8} & \frac{1}{\beta^2-\beta^9} & \frac{1}{\beta^2-\beta^{10}} & \frac{1}{\beta^2-\beta^{11}} \\ \frac{1}{\beta^3-\beta^8} & \frac{1}{\beta^3-\beta^9} & \frac{1}{\beta^3-\beta^{10}} & \frac{1}{\beta^3-\beta^{11}} \\ \hline \frac{1}{\beta^7-\beta^8} & \frac{1}{\beta^7-\beta^9} & \frac{1}{\beta^7-\beta^{10}} & \frac{1}{\beta^7-\beta^{11}} \end{array}\right] = \left[\begin{array}{ccc|c} \beta^5 & \beta^{12} & \beta^7 & \beta^9 \\ 1 & \beta^4 & \beta^{11} & \beta^6 \\ \beta^2 & \beta^{14} & \beta^3 & \beta^{10} \\ \hline \beta^4 & 1 & \beta^9 & \beta^7 \end{array}\right]$$

Example: Local Correction

- **Local erasure correction**

$$(1, \square, \beta^2, \quad \square, 0, 0) \qquad (\beta, 1, 0, \quad \beta^6, 0, \beta^{13})$$

$$\begin{matrix} (1,\beta,\beta^2) \\ \\ (\beta,1,0) \end{matrix} \quad \left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 \\ 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 \\ 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 \\ \hline 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 \\ 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} \\ 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 \end{array}\right]$$

$$\mathbf{T}_1 = \mathbf{T}_2 = \left[\begin{array}{c|c} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \hline \mathbf{U}_1 & \mathbf{Z}_1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \hline \mathbf{U}_2 & \mathbf{Z}_2 \end{array}\right] = \left[\begin{array}{ccc|c} \frac{1}{\beta-\beta^8} & \frac{1}{\beta-\beta^9} & \frac{1}{\beta-\beta^{10}} & \frac{1}{\beta-\beta^{11}} \\ \frac{1}{\beta^2-\beta^8} & \frac{1}{\beta^2-\beta^9} & \frac{1}{\beta^2-\beta^{10}} & \frac{1}{\beta^2-\beta^{11}} \\ \frac{1}{\beta^3-\beta^8} & \frac{1}{\beta^3-\beta^9} & \frac{1}{\beta^3-\beta^{10}} & \frac{1}{\beta^3-\beta^{11}} \\ \hline \frac{1}{\beta^7-\beta^8} & \frac{1}{\beta^7-\beta^9} & \frac{1}{\beta^7-\beta^{10}} & \frac{1}{\beta^7-\beta^{11}} \end{array}\right] = \left[\begin{array}{ccc|c} \beta^5 & \beta^{12} & \beta^7 & \beta^9 \\ 1 & \beta^4 & \beta^{11} & \beta^6 \\ \beta^2 & \beta^{14} & \beta^3 & \beta^{10} \\ \hline \beta^4 & 1 & \beta^9 & \beta^7 \end{array}\right]$$

Example: Local Correction

- **Local parity-check equations**

$$(1, \square, \beta^2, \quad \square, 0, 0) \qquad (\beta, 1, 0, \qquad \beta^6, 0, \beta^{13})$$

$$\begin{matrix} (1, \beta, \beta^2) \\ \\ (\beta, 1, 0) \end{matrix} \left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 \\ 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 \\ 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 \\ \hline 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 \\ 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} \\ 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 \end{array}\right]$$

$$\mathbf{T}_1 = \mathbf{T}_2 = \left[\begin{array}{c|c} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \hline \mathbf{U}_1 & \mathbf{Z}_1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \hline \mathbf{U}_2 & \mathbf{Z}_2 \end{array}\right] = \left[\begin{array}{ccc|c} \frac{1}{\beta-\beta^8} & \frac{1}{\beta-\beta^9} & \frac{1}{\beta-\beta^{10}} & \frac{1}{\beta-\beta^{11}} \\ \frac{1}{\beta^2-\beta^8} & \frac{1}{\beta^2-\beta^9} & \frac{1}{\beta^2-\beta^{10}} & \frac{1}{\beta^2-\beta^{11}} \\ \frac{1}{\beta^3-\beta^8} & \frac{1}{\beta^3-\beta^9} & \frac{1}{\beta^3-\beta^{10}} & \frac{1}{\beta^3-\beta^{11}} \\ \hline \frac{1}{\beta^7-\beta^8} & \frac{1}{\beta^7-\beta^9} & \frac{1}{\beta^7-\beta^{10}} & \frac{1}{\beta^7-\beta^{11}} \end{array}\right] = \left[\begin{array}{ccc|c} \beta^5 & \beta^{12} & \beta^7 & \beta^9 \\ 1 & \beta^4 & \beta^{11} & \beta^6 \\ \beta^2 & \beta^{14} & \beta^3 & \beta^{10} \\ \hline \beta^4 & 1 & \beta^9 & \beta^7 \end{array}\right]$$

$$(1, e_1, \beta^2, e_2, 0, 0, e_3) \begin{bmatrix} \beta^5 & \beta^{12} & \beta^7 \\ 1 & \beta^4 & \beta^{11} \\ \beta^2 & \beta^{14} & \beta^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \beta^4 & 1 & \beta^9 \end{bmatrix}$$

Example: Local Correction

- **Local parity-check equations**

$$(1, \square, \beta^2, \square, 0, 0) \quad (\beta, 1, 0, \beta^6, 0, \beta^{13})$$

$$\begin{array}{c} (1,\beta,\beta^2) \\ \\ (\beta,1,0) \end{array}
\left[\begin{array}{ccc|ccc|ccc|ccc}
1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 \\
0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 \\
0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 \\
\hline
0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 \\
0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} \\
0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3
\end{array}\right]$$

$$\mathbf{T}_1 = \mathbf{T}_2 = \left[\begin{array}{c|c} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \hline \mathbf{U}_1 & \mathbf{Z}_1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \hline \mathbf{U}_2 & \mathbf{Z}_2 \end{array}\right] = \left[\begin{array}{ccc|c}
\frac{1}{\beta-\beta^8} & \frac{1}{\beta-\beta^9} & \frac{1}{\beta-\beta^{10}} & \frac{1}{\beta-\beta^{11}} \\
\frac{1}{\beta^2-\beta^8} & \frac{1}{\beta^2-\beta^9} & \frac{1}{\beta^2-\beta^{10}} & \frac{1}{\beta^2-\beta^{11}} \\
\frac{1}{\beta^3-\beta^8} & \frac{1}{\beta^3-\beta^9} & \frac{1}{\beta^3-\beta^{10}} & \frac{1}{\beta^3-\beta^{11}} \\
\hline
\frac{1}{\beta^7-\beta^8} & \frac{1}{\beta^7-\beta^9} & \frac{1}{\beta^7-\beta^{10}} & \frac{1}{\beta^7-\beta^{11}}
\end{array}\right] = \left[\begin{array}{ccc|c}
\beta^5 & \beta^{12} & \beta^7 & \beta^9 \\
1 & \beta^4 & \beta^{11} & \beta^6 \\
\beta^2 & \beta^{14} & \beta^3 & \beta^{10} \\
\hline
\beta^4 & 1 & \beta^9 & \beta^7
\end{array}\right]$$

$$(1, e_1, \beta^2, e_2, 0, 0, e_3) \begin{bmatrix}
\beta^5 & \beta^{12} & \beta^7 \\
1 & \beta^4 & \beta^{11} \\
\beta^2 & \beta^{14} & \beta^3 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\beta^4 & 1 & \beta^9
\end{bmatrix}$$

Example: Local Correction

- **Local parity-check equations**

$$(1, \square, \beta^2, \square, 0, 0) \quad (\beta, 1, 0, \beta^6, 0, \beta^{13})$$

$$\begin{matrix} (1,\beta,\beta^2) \\ \\ (\beta,1,0) \end{matrix} \left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 \\ 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 \\ 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 \\ \hline 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 \\ 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} \\ 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 \end{array}\right]$$

$$\mathbf{T}_1 = \mathbf{T}_2 = \left[\begin{array}{c|c} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \hline \mathbf{U}_1 & \mathbf{Z}_1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \hline \mathbf{U}_2 & \mathbf{Z}_2 \end{array}\right] = \left[\begin{array}{ccc|c} \frac{1}{\beta-\beta^8} & \frac{1}{\beta-\beta^9} & \frac{1}{\beta-\beta^{10}} & \frac{1}{\beta-\beta^{11}} \\ \frac{1}{\beta^2-\beta^8} & \frac{1}{\beta^2-\beta^9} & \frac{1}{\beta^2-\beta^{10}} & \frac{1}{\beta^2-\beta^{11}} \\ \frac{1}{\beta^3-\beta^8} & \frac{1}{\beta^3-\beta^9} & \frac{1}{\beta^3-\beta^{10}} & \frac{1}{\beta^3-\beta^{11}} \\ \hline \frac{1}{\beta^7-\beta^8} & \frac{1}{\beta^7-\beta^9} & \frac{1}{\beta^7-\beta^{10}} & \frac{1}{\beta^7-\beta^{11}} \end{array}\right] = \left[\begin{array}{ccc|c} \beta^5 & \beta^{12} & \beta^7 & \beta^9 \\ 1 & \beta^4 & \beta^{11} & \beta^6 \\ \beta^2 & \beta^{14} & \beta^3 & \beta^{10} \\ \hline \beta^4 & 1 & \beta^9 & \beta^7 \end{array}\right]$$

$$(1, e_1, \beta^2, e_2, 0, 0, e_3) \begin{bmatrix} \beta^5 & \beta^{12} & \beta^7 \\ 1 & \beta^4 & \beta^{11} \\ \beta^2 & \beta^{14} & \beta^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \beta^4 & 1 & \beta^9 \end{bmatrix}$$

Example: Local Correction

- **Local parity-check equations**

$$(1,\beta,\beta^2,\quad \beta^{14},0,0) \qquad (\beta,1,0,\quad \beta^6,0,\beta^{13})$$

$$\begin{matrix}(1,\beta,\beta^2)\\ \\ (\beta,1,0)\end{matrix}\left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 \\ 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 \\ 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 \\ \hline 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 \\ 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} \\ 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 \end{array}\right]$$

$$\mathbf{T}_1 = \mathbf{T}_2 = \left[\begin{array}{c|c} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \hline \mathbf{U}_1 & \mathbf{Z}_1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \hline \mathbf{U}_2 & \mathbf{Z}_2 \end{array}\right] = \left[\begin{array}{ccc|c} \frac{1}{\beta-\beta^8} & \frac{1}{\beta-\beta^9} & \frac{1}{\beta-\beta^{10}} & \frac{1}{\beta-\beta^{11}} \\ \frac{1}{\beta^2-\beta^8} & \frac{1}{\beta^2-\beta^9} & \frac{1}{\beta^2-\beta^{10}} & \frac{1}{\beta^2-\beta^{11}} \\ \frac{1}{\beta^3-\beta^8} & \frac{1}{\beta^3-\beta^9} & \frac{1}{\beta^3-\beta^{10}} & \frac{1}{\beta^3-\beta^{11}} \\ \hline \frac{1}{\beta^7-\beta^8} & \frac{1}{\beta^7-\beta^9} & \frac{1}{\beta^7-\beta^{10}} & \frac{1}{\beta^7-\beta^{11}} \end{array}\right] = \left[\begin{array}{ccc|c} \beta^5 & \beta^{12} & \beta^7 & \beta^9 \\ 1 & \beta^4 & \beta^{11} & \beta^6 \\ \beta^2 & \beta^{14} & \beta^3 & \beta^{10} \\ \hline \beta^4 & 1 & \beta^9 & \beta^7 \end{array}\right]$$

$$(1,e_1,\beta^2,e_2,0,0,e_3)\begin{bmatrix} \beta^5 & \beta^{12} & \beta^7 \\ 1 & \beta^4 & \beta^{11} \\ \beta^2 & \beta^{14} & \beta^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \beta^4 & 1 & \beta^9 \end{bmatrix}$$

$$(e_1,e_2,e_3) = (\beta,\beta^{14},\beta^7)$$

Example: Global Correction

- **Global erasure correction**

$$(\boxed{}, \beta^2, \boxed{}, 0) \qquad (\beta, 1, 0, \qquad \beta^6, 0, \beta^{13})$$

$$\begin{array}{c} (1,\beta,\beta^2) \\ \\ (\beta,1,0) \end{array} \left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 \\ 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 \\ 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 \\ \hline 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 \\ 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} \\ 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 \end{array}\right]$$

$$\mathbf{T}_1 = \mathbf{T}_2 = \left[\begin{array}{c|c} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \hline \mathbf{U}_1 & \mathbf{Z}_1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \hline \mathbf{U}_2 & \mathbf{Z}_2 \end{array}\right] = \left[\begin{array}{ccc|c} \frac{1}{\beta-\beta^8} & \frac{1}{\beta-\beta^9} & \frac{1}{\beta-\beta^{10}} & \frac{1}{\beta-\beta^{11}} \\ \frac{1}{\beta^2-\beta^8} & \frac{1}{\beta^2-\beta^9} & \frac{1}{\beta^2-\beta^{10}} & \frac{1}{\beta^2-\beta^{11}} \\ \frac{1}{\beta^3-\beta^8} & \frac{1}{\beta^3-\beta^9} & \frac{1}{\beta^3-\beta^{10}} & \frac{1}{\beta^3-\beta^{11}} \\ \hline \frac{1}{\beta^7-\beta^8} & \frac{1}{\beta^7-\beta^9} & \frac{1}{\beta^7-\beta^{10}} & \frac{1}{\beta^7-\beta^{11}} \end{array}\right] = \left[\begin{array}{ccc|c} \beta^5 & \beta^{12} & \beta^7 & \beta^9 \\ 1 & \beta^4 & \beta^{11} & \beta^6 \\ \beta^2 & \beta^{14} & \beta^3 & \beta^{10} \\ \hline \beta^4 & 1 & \beta^9 & \beta^7 \end{array}\right]$$

Example: Global Correction

- **Global erasure correction**

$$(\square, \beta^2, \square, 0) \quad (\beta, 1, 0, \quad \beta^6, 0, \beta^{13})$$

$$\begin{array}{c} (1,\beta,\beta^2) \\ \\ (\beta,1,0) \end{array} \left[\begin{array}{ccc|ccc|ccc|ccc}
1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 \\
0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 \\
0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 \\
\hline
0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 \\
0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} \\
0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3
\end{array}\right]$$

$$\mathbf{T}_1 = \mathbf{T}_2 = \left[\begin{array}{c|c} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \hline \mathbf{U}_1 & \mathbf{Z}_1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \hline \mathbf{U}_2 & \mathbf{Z}_2 \end{array}\right] = \left[\begin{array}{ccc|c}
\frac{1}{\beta-\beta^8} & \frac{1}{\beta-\beta^9} & \frac{1}{\beta-\beta^{10}} & \frac{1}{\beta-\beta^{11}} \\
\frac{1}{\beta^2-\beta^8} & \frac{1}{\beta^2-\beta^9} & \frac{1}{\beta^2-\beta^{10}} & \frac{1}{\beta^2-\beta^{11}} \\
\frac{1}{\beta^3-\beta^8} & \frac{1}{\beta^3-\beta^9} & \frac{1}{\beta^3-\beta^{10}} & \frac{1}{\beta^3-\beta^{11}} \\
\hline
\frac{1}{\beta^7-\beta^8} & \frac{1}{\beta^7-\beta^9} & \frac{1}{\beta^7-\beta^{10}} & \frac{1}{\beta^7-\beta^{11}}
\end{array}\right] = \left[\begin{array}{ccc|c}
\beta^5 & \beta^{12} & \beta^7 & \beta^9 \\
1 & \beta^4 & \beta^{11} & \beta^6 \\
\beta^2 & \beta^{14} & \beta^3 & \beta^{10} \\
\hline
\beta^4 & 1 & \beta^9 & \beta^7
\end{array}\right]$$

$$\begin{aligned} & (e_1, e_2, \beta^2, e_3, e_4, 0) \\ - & (0, 0, 0, \beta^{11}, \beta^7, \beta) \\ = & (e_1', e_2', \beta^2, e_3', e_4', \beta) \end{aligned}$$

Example: Global Correction

- **Global erasure correction**

$$(\quad, \beta^2, \quad\quad, 0) \qquad (\beta, 1, 0, \quad \beta^6, 0, \beta^{13})$$

$$\begin{array}{c} (1,\beta,\beta^2) \\ \\ (\beta,1,0) \end{array} \left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 \\ 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 \\ 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 \\ \hline & & & & & & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 \\ & & & & & & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} \\ & & & & & & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 \end{array}\right]$$

$$\mathbf{T}_1 = \mathbf{T}_2 = \left[\begin{array}{c|c} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \hline \mathbf{U}_1 & \mathbf{Z}_1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \hline \mathbf{U}_2 & \mathbf{Z}_2 \end{array}\right] = \left[\begin{array}{ccc|c} \frac{1}{\beta-\beta^8} & \frac{1}{\beta-\beta^9} & \frac{1}{\beta-\beta^{10}} & \frac{1}{\beta-\beta^{11}} \\ \frac{1}{\beta^2-\beta^8} & \frac{1}{\beta^2-\beta^9} & \frac{1}{\beta^2-\beta^{10}} & \frac{1}{\beta^2-\beta^{11}} \\ \frac{1}{\beta^3-\beta^8} & \frac{1}{\beta^3-\beta^9} & \frac{1}{\beta^3-\beta^{10}} & \frac{1}{\beta^3-\beta^{11}} \\ \hline \frac{1}{\beta^7-\beta^8} & \frac{1}{\beta^7-\beta^9} & \frac{1}{\beta^7-\beta^{10}} & \frac{1}{\beta^7-\beta^{11}} \end{array}\right] = \left[\begin{array}{ccc|c} \beta^5 & \beta^{12} & \beta^7 & \beta^9 \\ 1 & \beta^4 & \beta^{11} & \beta^6 \\ \beta^2 & \beta^{14} & \beta^3 & \beta^{10} \\ \hline \beta^4 & 1 & \beta^9 & \beta^7 \end{array}\right]$$

$$(e_1', e_2', \beta^2, e_3', e_4', \beta) \begin{bmatrix} \beta^5 & \beta^{12} & \beta^7 & \beta^9 \\ 1 & \beta^4 & \beta^{11} & \beta^6 \\ \beta^2 & \beta^{14} & \beta^3 & \beta^{10} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} & (e_1, e_2, \beta^2, e_3, e_4, 0) \\ - & \\ = & (e_1', e_2', \beta^2, e_3', e_4', \beta) \end{aligned}$$

Example

- **Global erasure correction**

$$(\square, \beta^2, \square, 0) \qquad (\beta, 1, 0, \quad \beta^6, 0, \beta^{13})$$

$$\begin{array}{c} (1, \beta, \beta^2) \\ \\ (\beta, 1, 0) \end{array} \quad \left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 \\ 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 \\ 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 \\ \hline 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 \\ 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} \\ 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 \end{array}\right]$$

$$\mathbf{T}_1 = \mathbf{T}_2 = \left[\begin{array}{c|c} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \hline \mathbf{U}_1 & \mathbf{Z}_1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \hline \mathbf{U}_2 & \mathbf{Z}_2 \end{array}\right] = \left[\begin{array}{ccc|c} \frac{1}{\beta-\beta^8} & \frac{1}{\beta-\beta^9} & \frac{1}{\beta-\beta^{10}} & \frac{1}{\beta-\beta^{11}} \\ \frac{1}{\beta^2-\beta^8} & \frac{1}{\beta^2-\beta^9} & \frac{1}{\beta^2-\beta^{10}} & \frac{1}{\beta^2-\beta^{11}} \\ \frac{1}{\beta^3-\beta^8} & \frac{1}{\beta^3-\beta^9} & \frac{1}{\beta^3-\beta^{10}} & \frac{1}{\beta^3-\beta^{11}} \\ \hline \frac{1}{\beta^7-\beta^8} & \frac{1}{\beta^7-\beta^9} & \frac{1}{\beta^7-\beta^{10}} & \frac{1}{\beta^7-\beta^{11}} \end{array}\right] = \left[\begin{array}{ccc|c} \beta^5 & \beta^{12} & \beta^7 & \beta^9 \\ 1 & \beta^4 & \beta^{11} & \beta^6 \\ \beta^2 & \beta^{14} & \beta^3 & \beta^{10} \\ \hline \beta^4 & 1 & \beta^9 & \beta^7 \end{array}\right]$$

$$(e_1', e_2', \beta^2, e_3', e_4', \beta) \begin{bmatrix} \beta^5 & \beta^{12} & \beta^7 & \beta^9 \\ 1 & \beta^4 & \beta^{11} & \beta^6 \\ \beta^2 & \beta^{14} & \beta^3 & \beta^{10} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} & (e_1, e_2, \beta^2, e_3, e_4, 0) \\ - & (0, 0, 0, \beta^{11}, \beta^7, \beta) \\ = & (e_1', e_2', \beta^2, e_3', e_4', \beta) \end{aligned}$$

Example: Global Correction

- **Global erasure correction**

$$\left(\boxed{}, \beta^2, \boxed{}, 0\right) \qquad \left(\beta, 1, 0, \quad \beta^6, 0, \beta^{13}\right)$$

$$\begin{matrix} (1,\beta,\beta^2) \\ \\ (\beta,1,0) \end{matrix} \left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 \\ 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 \\ 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 \\ \hline 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 \\ 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} \\ 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 \end{array}\right]$$

$$\mathbf{T}_1 = \mathbf{T}_2 = \left[\begin{array}{c|c} \mathbf{A}_{1,1} & \mathbf{B}_{1,2} \\ \hline \mathbf{U}_1 & \mathbf{Z}_1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{A}_{2,2} & \mathbf{B}_{2,1} \\ \hline \mathbf{U}_2 & \mathbf{Z}_2 \end{array}\right] = \left[\begin{array}{ccc|c} \frac{1}{\beta-\beta^8} & \frac{1}{\beta-\beta^9} & \frac{1}{\beta-\beta^{10}} & \frac{1}{\beta-\beta^{11}} \\ \frac{1}{\beta^2-\beta^8} & \frac{1}{\beta^2-\beta^9} & \frac{1}{\beta^2-\beta^{10}} & \frac{1}{\beta^2-\beta^{11}} \\ \frac{1}{\beta^3-\beta^8} & \frac{1}{\beta^3-\beta^9} & \frac{1}{\beta^3-\beta^{10}} & \frac{1}{\beta^3-\beta^{11}} \\ \hline \frac{1}{\beta^7-\beta^8} & \frac{1}{\beta^7-\beta^9} & \frac{1}{\beta^7-\beta^{10}} & \frac{1}{\beta^7-\beta^{11}} \end{array}\right] = \left[\begin{array}{ccc|c} \beta^5 & \beta^{12} & \beta^7 & \beta^9 \\ 1 & \beta^4 & \beta^{11} & \beta^6 \\ \beta^2 & \beta^{14} & \beta^3 & \beta^{10} \\ \hline \beta^4 & 1 & \beta^9 & \beta^7 \end{array}\right]$$

$$(e_1', e_2', \beta^2, e_3', e_4', \beta) \begin{bmatrix} \beta^5 & \beta^{12} & \beta^7 & \beta^9 \\ 1 & \beta^4 & \beta^{11} & \beta^6 \\ \beta^2 & \beta^{14} & \beta^3 & \beta^{10} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} & (e_1, e_2, \beta^2, e_3, e_4, 0) \\ - & (0, 0, 0, \beta^{11}, \beta^7, \beta) \\ = & (e_1', e_2', \beta^2, e_3', e_4', \beta) \end{aligned}$$

$$(e_1', e_2', e_3', e_4') = (1, \beta, \beta^{10}, \beta^7)$$

Hierarchical Codes

- **Construction of hierarchical codes based on double-level codes**

$$\mathbf{G} = \left[\begin{array}{c|c|c|c} \mathbf{F}_{1,1} & \mathbf{F}_{1,2} & \cdots & \mathbf{F}_{1,p_0} \\ \hline \mathbf{F}_{2,1} & \mathbf{F}_{2,2} & \cdots & \mathbf{F}_{2,p_0} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{F}_{p_0,1} & \mathbf{F}_{p_0,2} & \cdots & \mathbf{F}_{p_0,p_0} \end{array}\right]$$

$$\mathbf{T}_{x,i} = \left[\begin{array}{c|c|c|c|c} \mathbf{A}_{x,x;i,i} & \mathbf{B}_{x,x;i} & \mathbf{E}_{x,1;i} & \cdots & \mathbf{E}_{x,p_0;i} \\ \hline \begin{array}{c} \mathbf{U}_{x,i} \\ \hline \mathbf{V}_{x,i} \end{array} & \multicolumn{4}{c}{\mathbf{Z}_{x,i}} \end{array}\right]$$

Hierarchical Codes

- **Construction of hierarchical codes based on double-level codes**
  - Local double-layer codes

$$\mathbf{G} = \begin{bmatrix} \mathbf{F}_{1,1} & \mathbf{F}_{1,2} & \cdots & \mathbf{F}_{1,p_0} \\ \mathbf{F}_{2,1} & \mathbf{F}_{2,2} & \cdots & \mathbf{F}_{2,p_0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{F}_{p_0,1} & \mathbf{F}_{p_0,2} & \cdots & \mathbf{F}_{p_0,p_0} \end{bmatrix}$$

Local double-layer codes

$$\mathbf{F}_{x,x} = \begin{bmatrix} \mathbf{I}_{k_{x,1}} & \mathbf{A}_{x,x;1,1} & \cdots & \mathbf{0} & \mathbf{A}_{x,x;1,p_x} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{A}_{x,x;p_x,1} & \cdots & \mathbf{I}_{k_{x,p_x}} & \mathbf{A}_{x,x;p_x,p_x} \end{bmatrix}$$

$$\mathbf{T}_{x,i} = \begin{bmatrix} \mathbf{A}_{x,x;i,i} & \mathbf{B}_{x,x;i} & \mathbf{E}_{x,1;i} & \cdots & \mathbf{E}_{x,p_0;i} \\ \mathbf{U}_{x,i} & & & & \\ \mathbf{V}_{x,i} & & \mathbf{Z}_{x,i} & & \end{bmatrix}$$

Hierarchical Codes

- **Construction of hierarchical codes based on double-level codes**
  - Local double-layer codes
    - $\mathbf{I}_{k_{x,i}}: k_{x,i} \times k_{x,i}$, $\mathbf{A}_{x,x;i,i}: k_{x,i} \times r_{x,i}$
    - $\mathbf{B}_{x,x;i,j}: k_{x,i} \times \delta_{y,j}$, $\mathbf{U}_{x,i}: \delta_{x,i} \times r_{x,i}$
    - $\mathbf{A}_{x,x;i,j} = \mathbf{B}_{x,x;i,j}\mathbf{U}_{x,j}$

$$\mathbf{G} = \left[\begin{array}{c|c|c|c} \mathbf{F}_{1,1} & \mathbf{F}_{1,2} & \cdots & \mathbf{F}_{1,p_0} \\ \hline \mathbf{F}_{2,1} & \mathbf{F}_{2,2} & \cdots & \mathbf{F}_{2,p_0} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{F}_{p_0,1} & \mathbf{F}_{p_0,2} & \cdots & \mathbf{F}_{p_0,p_0} \end{array}\right]$$

$$\mathbf{F}_{x,x} = \left[\begin{array}{c|c|c|c|c} \mathbf{I}_{k_{x,1}} & \mathbf{A}_{x,x;1,1} & \cdots & \mathbf{0} & \mathbf{A}_{x,x;1,p_x} \\ \hline \vdots & \ddots & \ddots & \vdots & \vdots \\ \hline \mathbf{0} & \mathbf{A}_{x,x;p_x,1} & \cdots & \mathbf{I}_{k_{x,p_x}} & \mathbf{A}_{x,x;p_x,p_x} \end{array}\right]$$

$$\mathbf{B}_{x,x;i} = \left[\begin{array}{c|c|c} \mathbf{B}_{x,x;i,1} & \cdots & \mathbf{B}_{x,x;i,p_x} \end{array}\right]$$

$$\mathbf{T}_{x,i} = \left[\begin{array}{c|c|c|c|c} \mathbf{A}_{x,x;i,i} & \mathbf{B}_{x,x;i} & \mathbf{E}_{x,1;i} & \cdots & \mathbf{E}_{x,p_0;i} \\ \hline \mathbf{U}_{x,i} & & & & \\ \hline \mathbf{V}_{x,i} & & \mathbf{Z}_{x,i} & & \end{array}\right]$$

Hierarchical Codes

- **Construction of hierarchical codes based on double-level codes**
  - Local double-layer codes
    - $\mathbf{I}_{k_{x,i}}: k_{x,i} \times k_{x,i}$, $\mathbf{A}_{x,x;i,i}: k_{x,i} \times r_{x,i}$
    - $\mathbf{B}_{x,x;i,j}: k_{x,i} \times \delta_{y,j}$, $\mathbf{U}_{x,i}: \delta_{x,i} \times r_{x,i}$
    - $\mathbf{A}_{x,x;i,j} = \mathbf{B}_{x,x;i,j}\mathbf{U}_{x,j}$
  - Second layer cross parities

$$\mathbf{G} = \begin{bmatrix} \mathbf{F}_{1,1} & \mathbf{F}_{1,2} & \cdots & \mathbf{F}_{1,p_0} \\ \mathbf{F}_{2,1} & \mathbf{F}_{2,2} & \cdots & \mathbf{F}_{2,p_0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{F}_{p_0,1} & \mathbf{F}_{p_0,2} & \cdots & \mathbf{F}_{p_0,p_0} \end{bmatrix}$$

Generate second layer cross parities

$$\mathbf{F}_{x,y} = \begin{bmatrix} \mathbf{0} & \mathbf{A}_{x,y;1,1} & \cdots & \mathbf{0} & \mathbf{A}_{x,y;1,p_y} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{A}_{x,y;p_x,1} & \cdots & \mathbf{0} & \mathbf{A}_{x,y;p_x,p_y} \end{bmatrix}$$

$$\mathbf{T}_{x,i} = \begin{bmatrix} \mathbf{A}_{x,x;i,i} & \mathbf{B}_{x,x;i} & \mathbf{E}_{x,1;i} & \cdots & \mathbf{E}_{x,p_0;i} \\ \mathbf{U}_{x,i} & & \mathbf{Z}_{x,i} & & \\ \mathbf{V}_{x,i} & & & & \end{bmatrix}$$

Hierarchical Codes

- **Construction of hierarchical codes based on double-level codes**
  - Local double-layer codes
    - $\mathbf{I}_{k_{x,i}}: k_{x,i} \times k_{x,i}$, $\mathbf{A}_{x,x;i,i}: k_{x,i} \times r_{x,i}$
    - $\mathbf{B}_{x,x;i,j}: k_{x,i} \times \delta_{y,j}$, $\mathbf{U}_{x,i}: \delta_{x,i} \times r_{x,i}$
    - $\mathbf{A}_{x,x;i,j} = \mathbf{B}_{x,x;i,j}\mathbf{U}_{x,j}$
  - Second layer cross parities
    - $\mathbf{V}_{x,i}: 2\gamma_x \times r_{x,i}$
    - $\mathbf{E}_{x,y;i,j}: k_{x,i} \times \gamma_y$
    - $\mathbf{E}_{x,y;i;p_y+1} = \mathbf{E}_{x,y;i;1}$
    - $\mathbf{A}_{x,y;i,j} = [\mathbf{E}_{x,y;i;j}, \mathbf{E}_{x,y;i;j+1}]\mathbf{V}_{y,j}$

$$\mathbf{G} = \left[\begin{array}{c|c|c|c} \mathbf{F}_{1,1} & \mathbf{F}_{1,2} & \cdots & \mathbf{F}_{1,p_0} \\ \hline \mathbf{F}_{2,1} & \mathbf{F}_{2,2} & \cdots & \mathbf{F}_{2,p_0} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{F}_{p_0,1} & \mathbf{F}_{p_0,2} & \cdots & \mathbf{F}_{p_0,p_0} \end{array}\right]$$

$$\mathbf{F}_{x,y} = \left[\begin{array}{c|c|c|c|c} \mathbf{0} & \mathbf{A}_{x,y;1,1} & \cdots & \mathbf{0} & \mathbf{A}_{x,y;1,p_y} \\ \hline \vdots & \ddots & \ddots & \vdots & \vdots \\ \hline \mathbf{0} & \mathbf{A}_{x,y;p_x,1} & \cdots & \mathbf{0} & \mathbf{A}_{x,y;p_x,p_y} \end{array}\right]$$

$$\mathbf{E}_{x,y;i} = \left[\begin{array}{c|c|c} \mathbf{E}_{x,y;i;1} & \cdots & \mathbf{E}_{x,y;i;p_y} \end{array}\right]$$

$$\mathbf{T}_{x,i} = \left[\begin{array}{c|c|c|c|c} \mathbf{A}_{x,x;i,i} & \mathbf{B}_{x,x;i} & \mathbf{E}_{x,1;i} & \cdots & \mathbf{E}_{x,p_0;i} \\ \hline \mathbf{U}_{x,i} & & & & \\ \mathbf{V}_{x,i} & \multicolumn{4}{c}{\mathbf{Z}_{x,i}} \end{array}\right]$$

Hierarchical Codes

- **1^{st} layer error correction capability**
  - $d_{1,x,i} = r_{x,i} - \delta_{x,i} - 2\gamma_x + 1$
- **2^{nd} layer error correction capability**
  - $d_{2,x,i} = r_{x,i} - \delta_{x,i} + \delta_x + 1$
  - $\delta_x = \delta_{x,1} + \cdots + \delta_{x,p_x}$
- **3^{rd} layer error correction capability**
  - $d_{3,x,i} = r_{x,i} - \delta_{x,i} + \delta_x - p_x\gamma_x + \gamma + 1$
  - $\gamma = p_1\gamma_1 + \cdots + p_{p_0}\gamma_{p_0}$

- Local double-layer codes
  - $\mathbf{I}_{k_{x,i}}: k_{x,i} \times k_{x,i}$, $\mathbf{A}_{x,x;i,i}: k_{x,i} \times r_{x,i}$
  - $\mathbf{B}_{x,x;i,j}: k_{x,i} \times \delta_{y,j}$, $\mathbf{U}_{x,i}: \delta_{x,i} \times r_{x,i}$
  - $\mathbf{A}_{x,x;i,j} = \mathbf{B}_{x,x;i,j}\mathbf{U}_{x,j}$
- Second layer cross parities
  - $\mathbf{V}_{x,i}: 2\gamma_x \times r_{x,i}$
  - $\mathbf{E}_{x,y;i,j}: k_{x,i} \times \gamma_y$
  - $\mathbf{E}_{x,y;i;p_y+1} = \mathbf{E}_{x,y;i;1}$
  - $\mathbf{A}_{x,y;i,j} = [\mathbf{E}_{x,y;i;j}, \mathbf{E}_{x,y;i;j+1}]\mathbf{V}_{y,j}$

$$\mathbf{T}_{x,i} = \left[\begin{array}{c|c|c|c|c} \mathbf{A}_{x,x;i,i} & \mathbf{B}_{x,x;i} & \mathbf{E}_{x,1;i} & \ldots & \mathbf{E}_{x,p_0;i} \\ \hline \mathbf{U}_{x,i} & \multicolumn{4}{c}{\multirow{2}{*}{\mathbf{Z}_{x,i}}} \\ \cline{1-1} \mathbf{V}_{x,i} & \multicolumn{4}{c}{} \end{array}\right]$$

Example

- **Example of a triple-level code**

$$
\left[\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc|ccc|ccc}
1 & 0 & 0 & \beta^{5} & \beta^{12} & \beta^{7} & 0 & 0 & 0 & \beta^{13} & \beta^{9} & \beta^{3} & 0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} \\
0 & 1 & 0 & 1 & \beta^{4} & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^{6} & 1 & 0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta & 0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta \\
0 & 0 & 1 & \beta^{2} & \beta^{14} & \beta^{3} & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^{4} & 0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} & 0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} \\
\hline
0 & 0 & 0 & \beta^{13} & \beta^{9} & \beta^{3} & 1 & 0 & 0 & \beta^{5} & \beta^{12} & \beta^{7} & 0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} \\
0 & 0 & 0 & \beta^{10} & \beta^{6} & 1 & 0 & 1 & 0 & 1 & \beta^{4} & \beta^{11} & 0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta & 0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta \\
0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^{4} & 0 & 0 & 1 & \beta^{2} & \beta^{14} & \beta^{3} & 0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} & 0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} \\
\hline
0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} & 1 & 0 & 0 & \beta^{5} & \beta^{12} & \beta^{7} & 0 & 0 & 0 & \beta^{13} & \beta^{9} & \beta^{3} \\
0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta & 0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta & 0 & 1 & 0 & 1 & \beta^{4} & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^{6} & 1 \\
0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} & 0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} & 0 & 0 & 1 & \beta^{2} & \beta^{14} & \beta^{3} & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^{4} \\
\hline
0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^{13} & \beta^{9} & \beta^{3} & 1 & 0 & 0 & \beta^{5} & \beta^{12} & \beta^{7} \\
0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta & 0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta & 0 & 0 & 0 & \beta^{10} & \beta^{6} & 1 & 0 & 1 & 0 & 1 & \beta^{4} & \beta^{11} \\
0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} & 0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^{4} & 0 & 0 & 1 & \beta^{2} & \beta^{14} & \beta^{3}
\end{array}\right]
$$

Example

- **Local double-level codes**

$$
\left[\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc|ccc|ccc}
1 & 0 & 0 & \beta^{5} & \beta^{12} & \beta^{7} & 0 & 0 & 0 & \beta^{13} & \beta^{9} & \beta^{3} & 0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} \\
0 & 1 & 0 & 1 & \beta^{4} & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^{6} & 1 & 0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta & 0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta \\
0 & 0 & 1 & \beta^{2} & \beta^{14} & \beta^{3} & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^{4} & 0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} & 0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} \\
\hline
0 & 0 & 0 & \beta^{13} & \beta^{9} & \beta^{3} & 1 & 0 & 0 & \beta^{5} & \beta^{12} & \beta^{7} & 0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} \\
0 & 0 & 0 & \beta^{10} & \beta^{6} & 1 & 0 & 1 & 0 & 1 & \beta^{4} & \beta^{11} & 0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta & 0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta \\
0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^{4} & 0 & 0 & 1 & \beta^{2} & \beta^{14} & \beta^{3} & 0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} & 0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} \\
\hline
0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} & 1 & 0 & 0 & \beta^{5} & \beta^{12} & \beta^{7} & 0 & 0 & 0 & \beta^{13} & \beta^{9} & \beta^{3} \\
0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta & 0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta & 0 & 1 & 0 & 1 & \beta^{4} & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^{6} & 1 \\
0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} & 0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} & 0 & 0 & 1 & \beta^{2} & \beta^{14} & \beta^{3} & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^{4} \\
\hline
0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^{13} & \beta^{9} & \beta^{3} & 1 & 0 & 0 & \beta^{5} & \beta^{12} & \beta^{7} \\
0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta & 0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta & 0 & 0 & 0 & \beta^{10} & \beta^{6} & 1 & 0 & 1 & 0 & 1 & \beta^{4} & \beta^{11} \\
0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} & 0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^{4} & 0 & 0 & 1 & \beta^{2} & \beta^{14} & \beta^{3}
\end{array}\right]
$$

Example

- **Second layer cross parities**

$$
\left[\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc|ccc|ccc}
1 & 0 & 0 & \beta^{5} & \beta^{12} & \beta^{7} & 0 & 0 & 0 & \beta^{13} & \beta^{9} & \beta^{3} & 0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} \\
0 & 1 & 0 & 1 & \beta^{4} & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^{6} & 1 & 0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta & 0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta \\
0 & 0 & 1 & \beta^{2} & \beta^{14} & \beta^{3} & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^{4} & 0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} & 0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} \\
\hline
0 & 0 & 0 & \beta^{13} & \beta^{9} & \beta^{3} & 1 & 0 & 0 & \beta^{5} & \beta^{12} & \beta^{7} & 0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} \\
0 & 0 & 0 & \beta^{10} & \beta^{6} & 1 & 0 & 1 & 0 & 1 & \beta^{4} & \beta^{11} & 0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta & 0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta \\
0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^{4} & 0 & 0 & 1 & \beta^{2} & \beta^{14} & \beta^{3} & 0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} & 0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} \\
\hline
0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} & 1 & 0 & 0 & \beta^{5} & \beta^{12} & \beta^{7} & 0 & 0 & 0 & \beta^{13} & \beta^{9} & \beta^{3} \\
0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta & 0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta & 0 & 1 & 0 & 1 & \beta^{4} & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^{6} & 1 \\
0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} & 0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} & 0 & 0 & 1 & \beta^{2} & \beta^{14} & \beta^{3} & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^{4} \\
\hline
0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^{3} & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^{13} & \beta^{9} & \beta^{3} & 1 & 0 & 0 & \beta^{5} & \beta^{12} & \beta^{7} \\
0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta & 0 & 0 & 0 & \beta^{9} & \beta^{3} & \beta & 0 & 0 & 0 & \beta^{10} & \beta^{6} & 1 & 0 & 1 & 0 & 1 & \beta^{4} & \beta^{11} \\
0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} & 0 & 0 & 0 & \beta^{6} & 1 & \beta^{13} & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^{4} & 0 & 0 & 1 & \beta^{2} & \beta^{14} & \beta^{3}
\end{array}\right]
$$

$$
\begin{bmatrix} \beta & \beta^{10} & \beta^{8} \end{bmatrix}
\begin{bmatrix} \beta^{2} \\ \beta^{8} \\ \beta^{5} \end{bmatrix}
=
\begin{bmatrix} \beta^{3} & \beta^{12} & \beta^{10} \\ \beta^{9} & \beta^{3} & \beta \\ \beta^{6} & 1 & \beta^{13} \end{bmatrix}
$$

Example

- **Messages and codewords**

$$(1,\beta,\beta^2,\beta^{12},\beta^{14},\beta^{12}) \quad (\beta,1,0,\beta^9,\beta^{14},\beta)$$

$$\begin{array}{c} (1,\beta,\beta^2) \\ \\ (\beta,1,0) \\ \\ (\beta^2,0,\beta) \\ \\ (0,\beta,1) \end{array}
\left[\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc|ccc|ccc}
1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} \\
0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 0 & 0 & \beta^9 & \beta^3 & \beta & 0 & 0 & 0 & \beta^9 & \beta^3 & \beta \\
0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 0 & \beta^6 & 1 & \beta^{13} & 0 & 0 & 0 & \beta^6 & 1 & \beta^{13} \\
\hline
0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} \\
0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^9 & \beta^3 & \beta & 0 & 0 & 0 & \beta^9 & \beta^3 & \beta \\
0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^6 & 1 & \beta^{13} & 0 & 0 & 0 & \beta^6 & 1 & \beta^{13} \\
\hline
0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 \\
0 & 0 & 0 & \beta^9 & \beta^3 & \beta & 0 & 0 & 0 & \beta^9 & \beta^3 & \beta & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 \\
0 & 0 & 0 & \beta^6 & 1 & \beta^{13} & 0 & 0 & 0 & \beta^6 & 1 & \beta^{13} & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 \\
\hline
0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 \\
0 & 0 & 0 & \beta^9 & \beta^3 & \beta & 0 & 0 & 0 & \beta^9 & \beta^3 & \beta & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} \\
0 & 0 & 0 & \beta^6 & 1 & \beta^{13} & 0 & 0 & 0 & \beta^6 & 1 & \beta^{13} & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3
\end{array}\right]$$

$$\begin{bmatrix} \beta & \beta^{10} & \beta^8 \end{bmatrix} \begin{bmatrix} \beta^2 \\ \beta^8 \\ \beta^5 \end{bmatrix} = \begin{bmatrix} \beta^3 & \beta^{12} & \beta^{10} \\ \beta^9 & \beta^3 & \beta \\ \beta^6 & 1 & \beta^{13} \end{bmatrix}$$

Example

- **Middle-layer erasure correction**

$$(\square\ \beta, \beta^2, \square, \beta^{12})\ (\beta, 1, 0, \beta^9, \beta^{14}, \beta)$$

$$\begin{array}{c} (1,\beta,\beta^2) \\ \\ (\beta,1,0) \\ \\ (\beta^2,0,\beta) \\ \\ (0,\beta,1) \end{array}
\left[\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} \\
0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 0 & 0 & \beta^9 & \beta^3 & \beta \\
0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 0 & \beta^6 & 1 & \beta^{13} \\
\hline
0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} \\
0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^9 & \beta^3 & \beta \\
0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^6 & 1 & \beta^{13} \\
\hline
0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 \\
0 & 0 & 0 & \beta^9 & \beta^3 & \beta & 0 & 0 & 0 & \beta^9 & \beta^3 & \beta & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} \\
0 & 0 & 0 & \beta^6 & 1 & \beta^{13} & 0 & 0 & 0 & \beta^6 & 1 & \beta^{13} & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 \\
\hline
0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 \\
0 & 0 & 0 & \beta^9 & \beta^3 & \beta & 0 & 0 & 0 & \beta^9 & \beta^3 & \beta & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 \\
0 & 0 & 0 & \beta^6 & 1 & \beta^{13} & 0 & 0 & 0 & \beta^6 & 1 & \beta^{13} & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4
\end{array}\right.$$

$$\left[\begin{array}{ccc} \beta & \beta^{10} & \beta^8 \end{array}\right] \begin{bmatrix} \beta^2 \\ \beta^8 \\ \beta^5 \end{bmatrix} = \begin{bmatrix} \beta^3 & \beta^{12} & \beta^{10} \\ \beta^9 & \beta^3 & \beta \\ \beta^6 & 1 & \beta^{13} \end{bmatrix}$$

Example

- **Middle-layer erasure correction**

$(\square, \beta, \beta^2, \square, \square, \beta^{12})$ $(\beta, 1, 0, \beta^9, \beta^{14}, \beta)$

	1	0	0	β^5	β^{12}	β^7	0	0	0	β^{13}	β^9	β^3	0	0	0	β^3	β^{12}	β^{10}	0	0	0	β^3	β^{12}	β^{10}
$(1, \beta, \beta^2)$	0	1	0	1	β^4	β^{11}	0	0	0	β^{10}	β^6	1	0	0	0	β^9	β^3	β	0	0	0	β^9	β^3	β
	0	0	1	β^2	β^{14}	β^3	0	0	0	β^{14}	β^{10}	β^4	0	0	0	β^6	1	β^{13}	0	0	0	β^6	1	β^{13}
	0	0	0	β^{13}	β^9	β^3	1	0	0	β^5	β^{12}	β^7	0	0	0	β^3	β^{12}	β^{10}	0	0	0	β^3	β^{12}	β^{10}
$(\beta, 1, 0)$	0	0	0	β^{10}	β^6	1	0	1	0	1	β^4	β^{11}	0	0	0	β^9	β^3	β	0	0	0	β^9	β^3	β
	0	0	0	β^{14}	β^{10}	β^4	0	0	1	β^2	β^{14}	β^3	0	0	0	β^6	1	β^{13}	0	0	0	β^6	1	β^{13}
	0	0	0	β^3	β^{12}	β^{10}	0	0	0	β^3	β^{12}	β^{10}	1	0	0	β^5	β^{12}	β^7	0	0	0	β^{13}	β^9	β^3
$(\beta^2, 0, \beta)$	0	0	0	β^9	β^3	β	0	0	0	β^9	β^3	β	0	1	0	1	β^4	β^{11}	0	0	0	β^{10}	β^6	1
	0	0	0	β^6	1	β^{13}	0	0	0	β^6	1	β^{13}	0	0	1	β^2	β^{14}	β^3	0	0	0	β^{14}	β^{10}	β^4
	0	0	0	β^3	β^{12}	β^{10}	0	0	0	[illegible]	[illegible]	[illegible]	[illegible]	0	0	β^{13}	β^9	β^3	1	0	0	β^5	β^{12}	β^7
$(0, \beta, 1)$	0	0	0	β^9	β^3	β	0	0	0	[illegible]	[illegible]	[illegible]	[illegible]	0	0	β^{10}	β^6	1	0	1	0	1	β^4	β^{11}
	0	0	0	β^6	1	β^{13}	0	0	0	β^6	1	β^{13}	0	0	0	β^{14}	β^{10}	β^4	0	0	1	β^2	β^{14}	β^3

$+ (\beta^{11}, \beta^7, \beta)$

$$\begin{bmatrix} \beta & \beta^{10} & \beta^8 \end{bmatrix} \begin{bmatrix} \beta^2 \\ \beta^8 \\ \beta^5 \end{bmatrix} = \begin{bmatrix} \beta^3 & \beta^{12} & \beta^{10} \\ \beta^9 & \beta^3 & \beta \\ \beta^6 & 1 & \beta^{13} \end{bmatrix}$$

$$\begin{aligned} &(e_1, \beta, \beta^2, e_2, e_3, \beta^{12}) \\ &+ (\beta^5, \beta^{14}, \beta^{12}) \\ &+ (\beta^{11}, \beta^7, \beta) \\ &= (e_1', \beta, \beta^2, e_2', e_3', \beta) \end{aligned}$$

Example

- **Middle-layer erasure correction**

$$(\square, \beta, \beta^2, \square\ \square, \beta^{12})\ (\beta, 1, 0, \beta^9, \beta^{14}, \beta)$$

$$\begin{matrix} (1,\beta,\beta^2) \\ \\ (\beta,1,0) \\ \\ (\beta^2,0,\beta) \\ \\ (0,\beta,1) \end{matrix}
\left[\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc|}
1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} \\
0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 0 & 0 & \beta^9 & \beta^3 & \beta & 0 & 0 & 0 & \beta^9 & \beta^3 & \beta \\
0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 0 & \beta^6 & 1 & \beta^{13} & 0 & 0 & 0 & \beta^6 & 1 & \beta^{13} \\
\hline
0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} \\
0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^9 & \beta^3 & \beta & 0 & 0 & 0 & \beta^9 & \beta^3 & \beta \\
0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^6 & 1 & \beta^{13} & 0 & 0 & 0 & \beta^6 & 1 & \beta^{13} \\
\hline
0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 \\
0 & 0 & 0 & \beta^9 & \beta^3 & \beta & 0 & 0 & 0 & \beta^9 & \beta^3 & \beta & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 \\
0 & 0 & 0 & \beta^6 & 1 & \beta^{13} & 0 & 0 & 0 & \beta^6 & 1 & \beta^{13} & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 \\
\hline
0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 \\
0 & 0 & 0 & \beta^9 & \beta^3 & \beta & 0 & 0 & 0 & \beta^9 & \beta^3 & \beta & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} \\
0 & 0 & 0 & \beta^6 & 1 & \beta^{13} & 0 & 0 & 0 & \beta^6 & 1 & \beta^{13} & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3
\end{array}\right]$$

$$\begin{bmatrix} \beta & \beta^{10} & \beta^8 \end{bmatrix} \begin{bmatrix} \beta^2 \\ \beta^8 \\ \beta^5 \end{bmatrix} = \begin{bmatrix} \beta^3 & \beta^{12} & \beta^{10} \\ \beta^9 & \beta^3 & \beta \\ \beta^6 & 1 & \beta^{13} \end{bmatrix}$$

$$\begin{aligned} &(e_1, \beta, \beta^2, e_2, e_3, \beta^{12}) \\ &+ (\beta^5, \beta^{14}, \beta^{12}) \\ &+ (\beta^{11}, \beta^7, \beta) \\ &= (e_1', \beta, \beta^2, e_2', e_3', \beta) \end{aligned}$$

Example

- **Middle-layer parity-check equations**

$$(\quad , \beta, \beta^2, \quad , \beta^{12})\,(\beta, 1, 0, \beta^9, \beta^{14}, \beta)$$

$(1,\beta,\beta^2)$	1	0	0	β^5	β^{12}	β^7	0	0	0	β^{13}	β^9	β^3	0	0	0	β^3	β^{12}	β^{10}	0	0	0	β^3	β^{12}	β^{10}
	0	1	0	1	β^4	β^{11}	0	0	0	β^{10}	β^6	1	0	0	0	β^9	β^3	β	0	0	0	β^9	β^3	β
	0	0	1	β^2	β^{14}	β^3	0	0	0	β^{14}	β^{10}	β^4	0	0	0	β^6	1	β^{13}	0	0	0	β^6	1	β^{13}
$(\beta,1,0)$	0	0	0				1	0	0				0	0	0	β^3	β^{12}	β^{10}	0	0	0	β^3	β^{12}	β^{10}
	0	0	0				0	1	0				0	0	0	β^9	β^3	β	0	0	0	β^9	β^3	β
	0	0	0				0	0	1				0	0	0	β^6	1	β^{13}	0	0	0	β^6	1	β^{13}
$(\beta^2,0,\beta)$	0	0	0				0	0	0				1	0	0	β^5	β^{12}	β^7	0	0	0	β^{13}	β^9	β^3
	0	0	0				0	0	0				0	1	0	1	β^4	β^{11}	0	0	0	β^{10}	β^6	1
	0	0	0				0	0	0				0	0	1	β^2	β^{14}	β^3	0	0	0	β^{14}	β^{10}	β^4
$(0,\beta,1)$	0	0	0				0	0	0				0	0	0	β^{13}	β^9	β^3	1	0	0	β^5	β^{12}	β^7
	0	0	0				0	0	0				0	0	0	β^{10}	β^6	1	0	1	0	1	β^4	β^{11}
	0	0	0				0	0	0				0	0	0	β^{14}	β^{10}	β^4	0	0	1	β^2	β^{14}	β^3

$$(e'_1, \beta, \beta^2, e'_2, e'_3, \beta)\begin{bmatrix} \beta^5 & \beta^{12} & \beta^7 & \beta^9 \\ 1 & \beta^4 & \beta^{11} & \beta^6 \\ \beta^2 & \beta^{14} & \beta^3 & \beta^{10} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \beta & \beta^{10} & \beta^8 \end{bmatrix}\begin{bmatrix} \beta^2 \\ \beta^8 \\ \beta^5 \end{bmatrix} = \begin{bmatrix} \beta^3 & \beta^{12} & \beta^{10} \\ \beta^9 & \beta^3 & \beta \\ \beta^6 & 1 & \beta^{13} \end{bmatrix}$$

$$(e_1, \beta, \beta^2, e_2, e_3, \beta^{12})$$

$$= (e'_1, \beta, \beta^2, e'_2, e'_3, \beta)$$

Example

- **Middle-layer parity-check equations**

$$(1,\beta,\beta^2,\beta^{12},\beta^{14},\beta^{12})\ (\beta,1,0,\beta^9,\beta^{14},\beta)$$

$$\begin{array}{c} (1,\beta,\beta^2) \\ \\ (\beta,1,0) \\ \\ (\beta^2,0,\beta) \\ \\ (0,\beta,1) \end{array}
\left[\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc|ccc|ccc}
1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} \\
0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 0 & 0 & \beta^9 & \beta^3 & \beta & 0 & 0 & 0 & \beta^9 & \beta^3 & \beta \\
0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 0 & \beta^6 & 1 & \beta^{13} & 0 & 0 & 0 & \beta^6 & 1 & \beta^{13} \\
\hline
0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} \\
0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^9 & \beta^3 & \beta & 0 & 0 & 0 & \beta^9 & \beta^3 & \beta \\
0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^6 & 1 & \beta^{13} & 0 & 0 & 0 & \beta^6 & 1 & \beta^{13} \\
\hline
0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 \\
0 & 0 & 0 & \beta^9 & \beta^3 & \beta & 0 & 0 & 0 & \beta^9 & \beta^3 & \beta & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 \\
0 & 0 & 0 & \beta^6 & 1 & \beta^{13} & 0 & 0 & 0 & \beta^6 & 1 & \beta^{13} & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3 & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 \\
\hline
0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^3 & \beta^{12} & \beta^{10} & 0 & 0 & 0 & \beta^{13} & \beta^9 & \beta^3 & 1 & 0 & 0 & \beta^5 & \beta^{12} & \beta^7 \\
0 & 0 & 0 & \beta^9 & \beta^3 & \beta & 0 & 0 & 0 & \beta^9 & \beta^3 & \beta & 0 & 0 & 0 & \beta^{10} & \beta^6 & 1 & 0 & 1 & 0 & 1 & \beta^4 & \beta^{11} \\
0 & 0 & 0 & \beta^6 & 1 & \beta^{13} & 0 & 0 & 0 & \beta^6 & 1 & \beta^{13} & 0 & 0 & 0 & \beta^{14} & \beta^{10} & \beta^4 & 0 & 0 & 1 & \beta^2 & \beta^{14} & \beta^3
\end{array}\right]$$

$$(e_1',\beta,\beta^2,e_2',e_3',\beta)\begin{bmatrix} \beta^5 & \beta^{12} & \beta^7 & \beta^9 \\ 1 & \beta^4 & \beta^{11} & \beta^6 \\ \beta^2 & \beta^{14} & \beta^3 & \beta^{10} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$(e_1',e_2',e_3') = (1,\beta^{10},\beta^7)$$

$$\begin{bmatrix} \beta & \beta^{10} & \beta^8 \end{bmatrix}\begin{bmatrix} \beta^2 \\ \beta^8 \\ \beta^5 \end{bmatrix} = \begin{bmatrix} \beta^3 & \beta^{12} & \beta^{10} \\ \beta^9 & \beta^3 & \beta \\ \beta^6 & 1 & \beta^{13} \end{bmatrix}$$

$$\begin{aligned} &(e_1,\beta,\beta^2,e_2,e_3,\beta^{12}) \\ &+(\beta^5,\beta^{14},\beta^{12}) \\ &+(\beta^{11},\beta^7,\beta) \\ &=(e_1',\beta,\beta^2,e_2',e_3',\beta) \end{aligned}$$

Heterogeneity

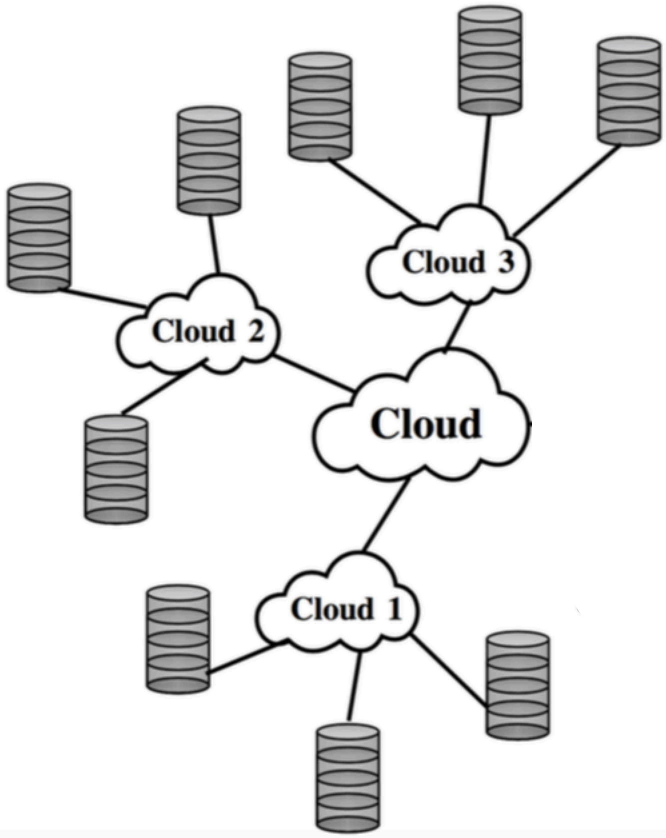
- **Parameters**
  - $(n_{1,1}, n_{1,2}, n_{2,1}, n_{2,2}) = (10,11,10,10)$
  - $(n_{3,1}, n_{3,2}, n_{3,3}, n_{3,4}) = (12,12,12,12)$
  - $(k_{1,1}, k_{1,2}, k_{2,1}, k_{2,2}) = (6,6,7,7)$
  - $(k_{3,1}, k_{3,2}, k_{3,3}, k_{3,4}) = (9,8,9,9)$
  - $(r_{1,1}, r_{1,2}, r_{2,1}, r_{2,2}) = (4,5,3,3)$
  - $(r_{3,1}, r_{3,2}, r_{3,3}, r_{3,4}) = (3,4,3,3)$
  - $(\delta_{1,1}, \delta_{1,2}) = (\delta_{2,1}, \delta_{2,2}) = (1,1)$
  - $\delta_1 = \delta_2 = \delta_{1,1} + \delta_{1,2} = 2$
  - $(\delta_{3,1}, \delta_{3,2}, \delta_{3,3}, \delta_{3,4}) = (1,2,1,1)$
  - $\delta_3 = 1 + 2 + 1 + 1 = 5$
  - $(\gamma_1, \gamma_2, \gamma_3) = (1, ½, ½)$
  - $\gamma = 2 \cdot (1) + 2 \cdot (½) + 4 \cdot (½) = 5$

- **1st layer error correction capability**
  - $d_{1,x,i} = r_{x,i} - \delta_{x,i} - 2\gamma_x + 1$
- **2nd layer error correction capability**
  - $d_{2,x,i} = r_{x,i} - \delta_{x,i} + \delta_x + 1$
  - $\delta_x = \delta_{x,1} + \cdots + \delta_{x,p_x}$
- **3rd layer error correction capability**
  - $d_{3,x,i} = r_{x,i} - \delta_{x,i} + \delta_x - p_x\gamma_x + \gamma + 1$
  - $\gamma = p_1\gamma_1 + \cdots + p_{p_0}\gamma_{p_0}$

- **Unequal error protection**

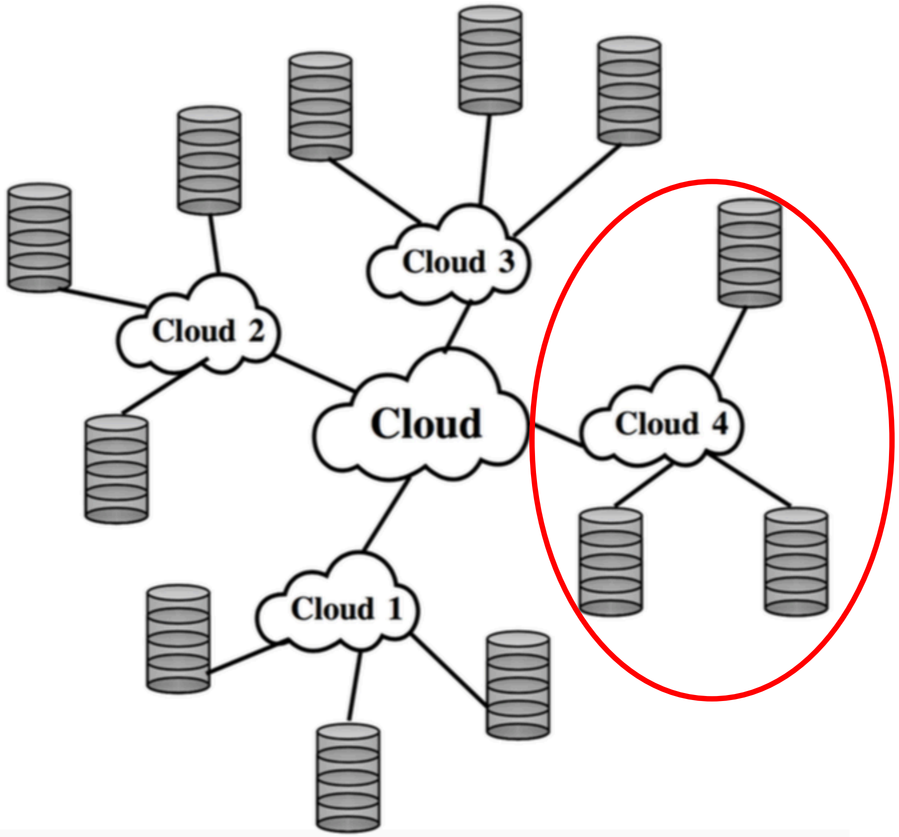
$$\left[\begin{array}{cc|cc|cccc} 2 & 3 & 2 & 2 & 2 & 2 & 2 & 2 \\ \hline 6 & 7 & 5 & 5 & 8 & 8 & 8 & 8 \\ \hline 9 & 10 & 9 & 9 & 11 & 11 & 11 & 11 \end{array}\right]$$

Scalability



Scalability

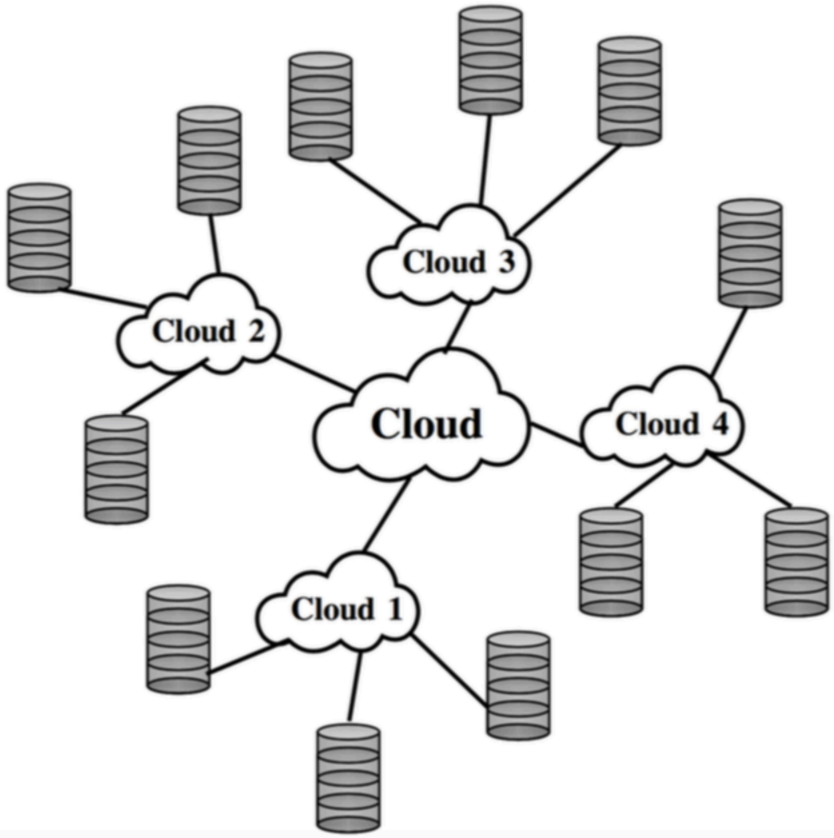
- **Add a cloud to the existing network**



Scalability

- **Add a cloud to the existing network**
  - **Step 1**: parameter selection
    - Cloud 4 chooses its local parameters

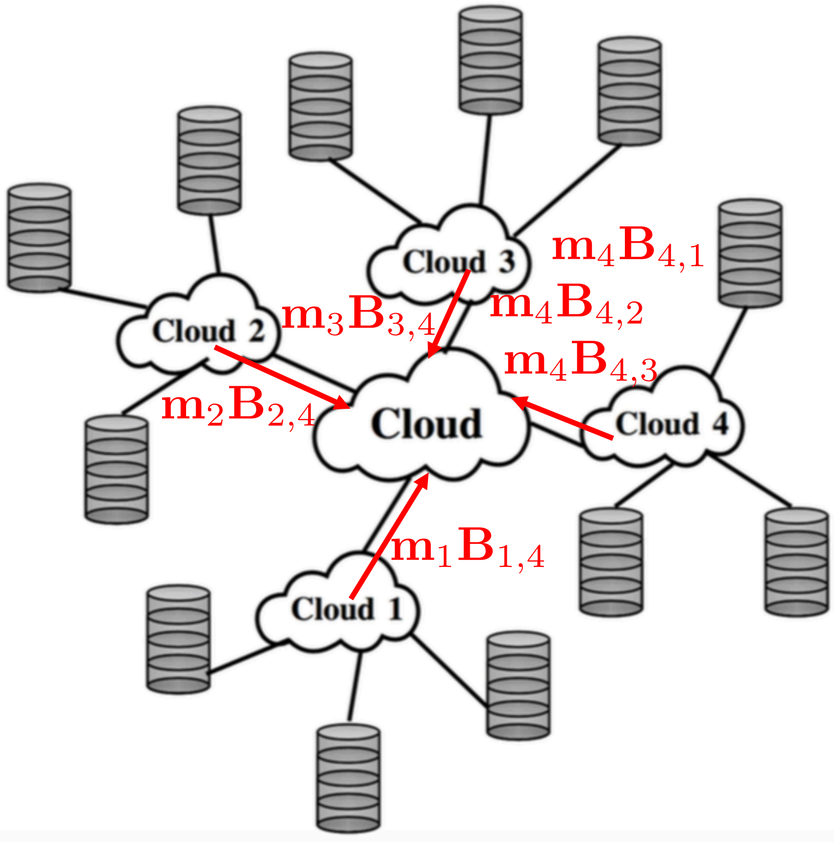
$$\mathbf{T}_4 = \left[\begin{array}{c|c|c|c} \mathbf{A}_{4,4} & \mathbf{B}_{4,1} & \mathbf{B}_{4,2} & \mathbf{B}_{4,3} \\ \hline \mathbf{U}_4 & & & \end{array}\right]$$



Scalability

- **Add a cloud to the existing network**
  - **Step 1**: parameter selection
    - Cloud 4 chooses its local parameters
  - **Step 2**: information exchange
    - Uplink

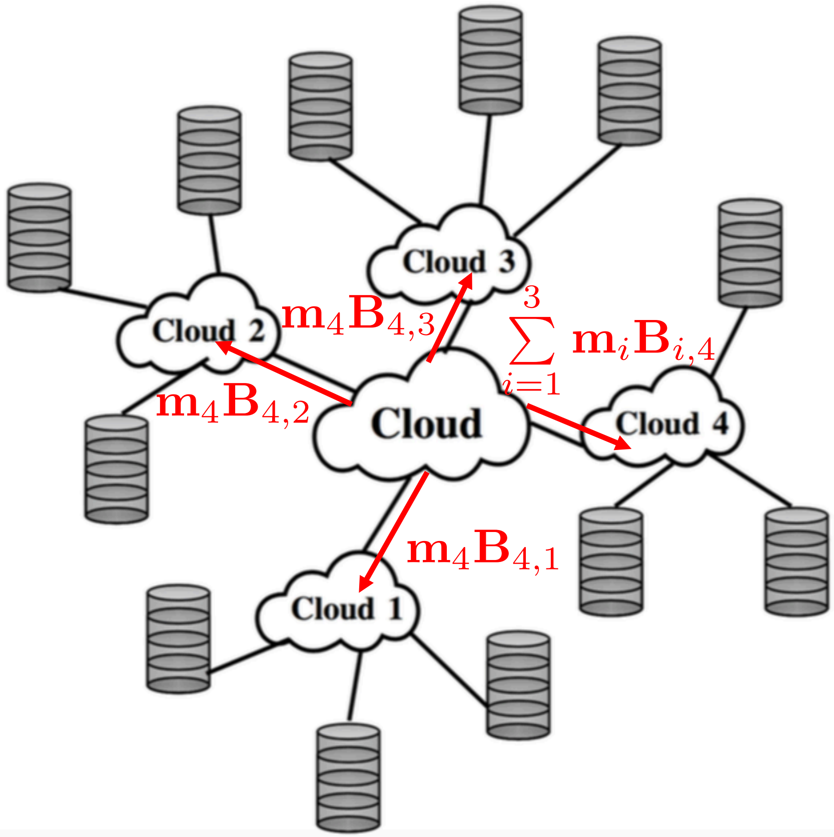
$$\mathbf{T}_4 = \left[\begin{array}{c|c|c|c} \mathbf{A}_{4,4} & \mathbf{B}_{4,1} & \mathbf{B}_{4,2} & \mathbf{B}_{4,3} \\ \hline \mathbf{U}_4 & & & \end{array}\right]$$



Scalability

- **Add a cloud to the existing network**
  - **Step 1**: parameter selection
    - Cloud 4 chooses its local parameters
  - **Step 2**: information exchange
    - Uplink
    - Downlink

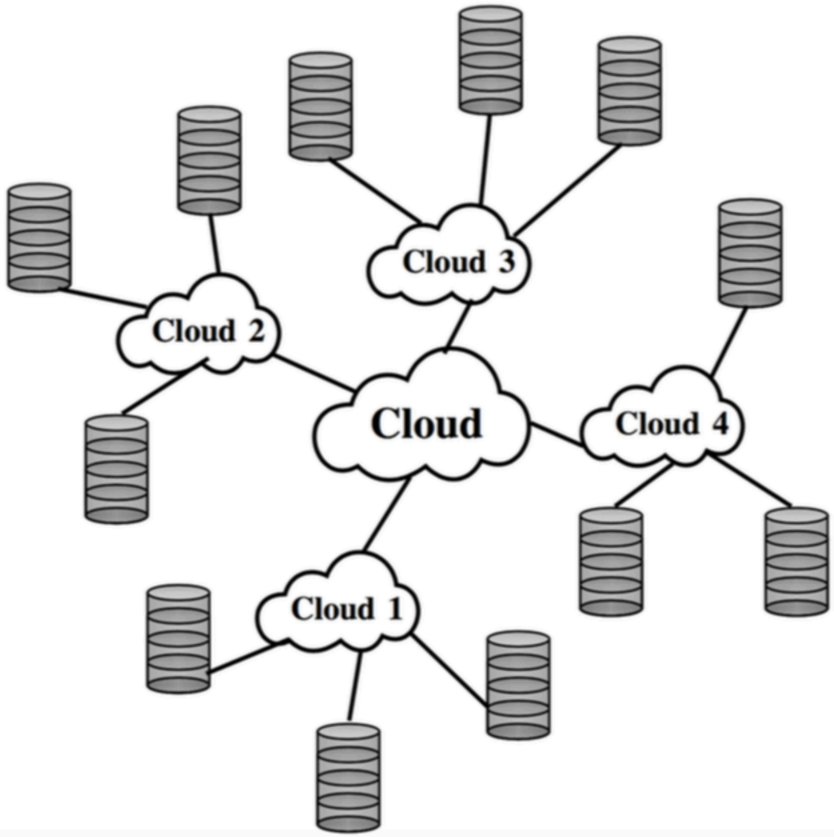
$$\mathbf{T}_4 = \left[\begin{array}{c|c|c|c} \mathbf{A}_{4,4} & \mathbf{B}_{4,1} & \mathbf{B}_{4,2} & \mathbf{B}_{4,3} \\ \hline \mathbf{U}_4 & & & \end{array}\right]$$



Scalability

- **Add a cloud to the existing network**
  - **Step 1**: parameter selection
    - Cloud 4 chooses its local parameters
  - **Step 2**: information exchange
    - Uplink
    - Downlink
  - **Step 3**: update
    - Local cloud i adds $\mathbf{m}_4 \mathbf{B}_{4,i} \mathbf{U}_i$ to its original parity
    - Local cloud 4 computes its parity $\mathbf{m}_4 \mathbf{A}_{4,4} + \sum_{i=1}^{3} \mathbf{m}_i \mathbf{B}_{i,4} \mathbf{U}_4$

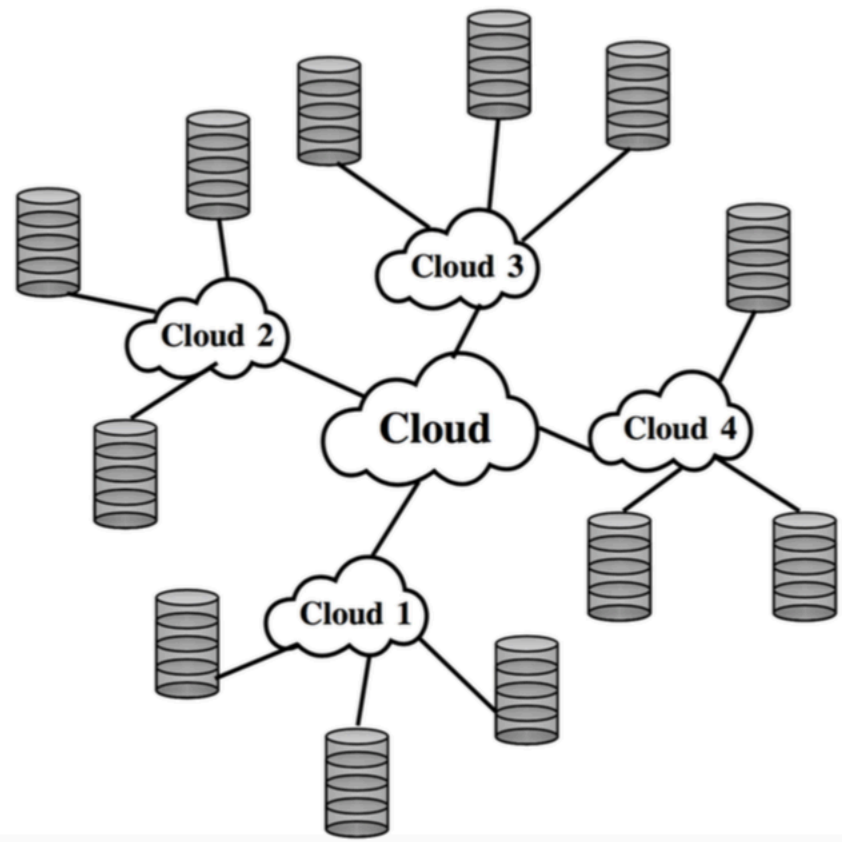
$$\mathbf{T}_4 = \left[\begin{array}{c|c|c|c} \mathbf{A}_{4,4} & \mathbf{B}_{4,1} & \mathbf{B}_{4,2} & \mathbf{B}_{4,3} \\ \hline \mathbf{U}_4 & & & \end{array} \right]$$



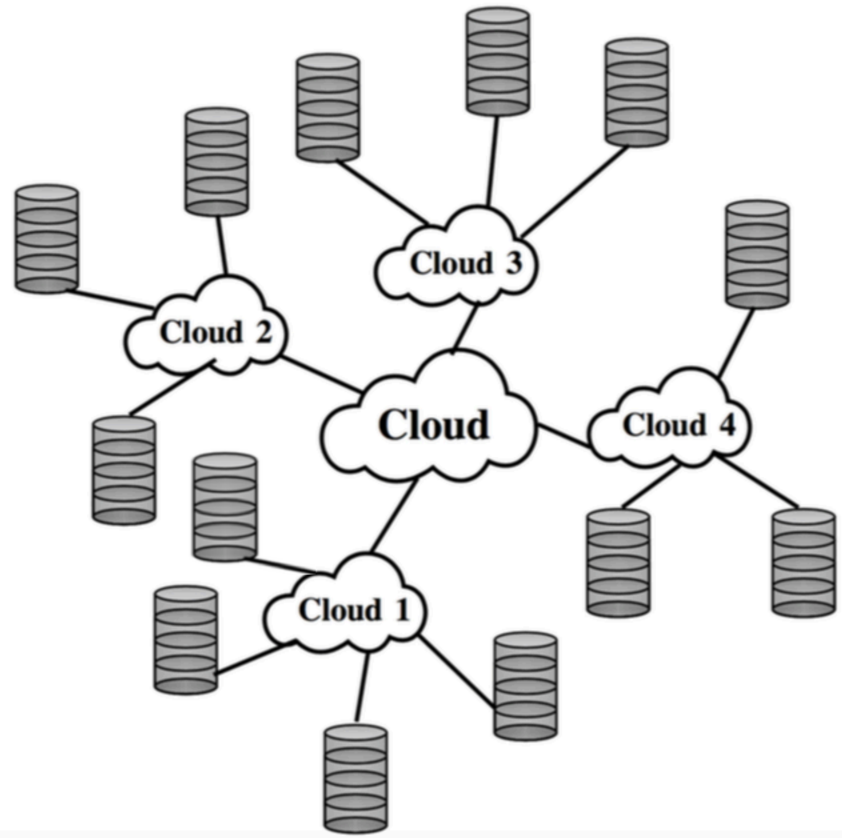
Scalability

- **Add a cloud to the existing network**
  - **Step 1**: parameter selection
    - Cloud 4 chooses its local parameters
  - **Step 2**: information exchange
    - Uplink
    - Downlink
  - **Step 3**: update
    - Local cloud i adds $\mathbf{m}_4\mathbf{B}_{4,i}\mathbf{U}_i$ to its original parity
    - Local cloud 4 computes its parity $\mathbf{m}_4\mathbf{A}_{4,4} + \sum_{i=1}^{3}\mathbf{m}_i\mathbf{B}_{i,4}\mathbf{U}_4$
  - **Local parameters of other clouds remain unchanged**

$$\mathbf{T}_4 = \left[\begin{array}{c|c|c|c} \mathbf{A}_{4,4} & \mathbf{B}_{4,1} & \mathbf{B}_{4,2} & \mathbf{B}_{4,3} \\ \hline \mathbf{U}_4 & & & \end{array}\right]$$

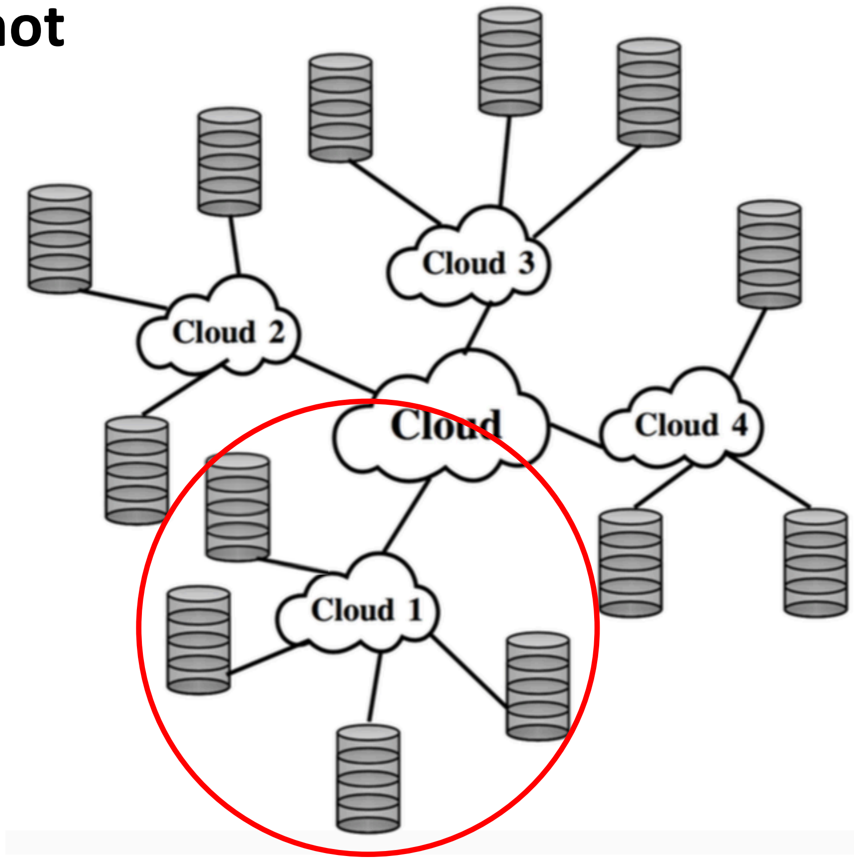


Flexibility



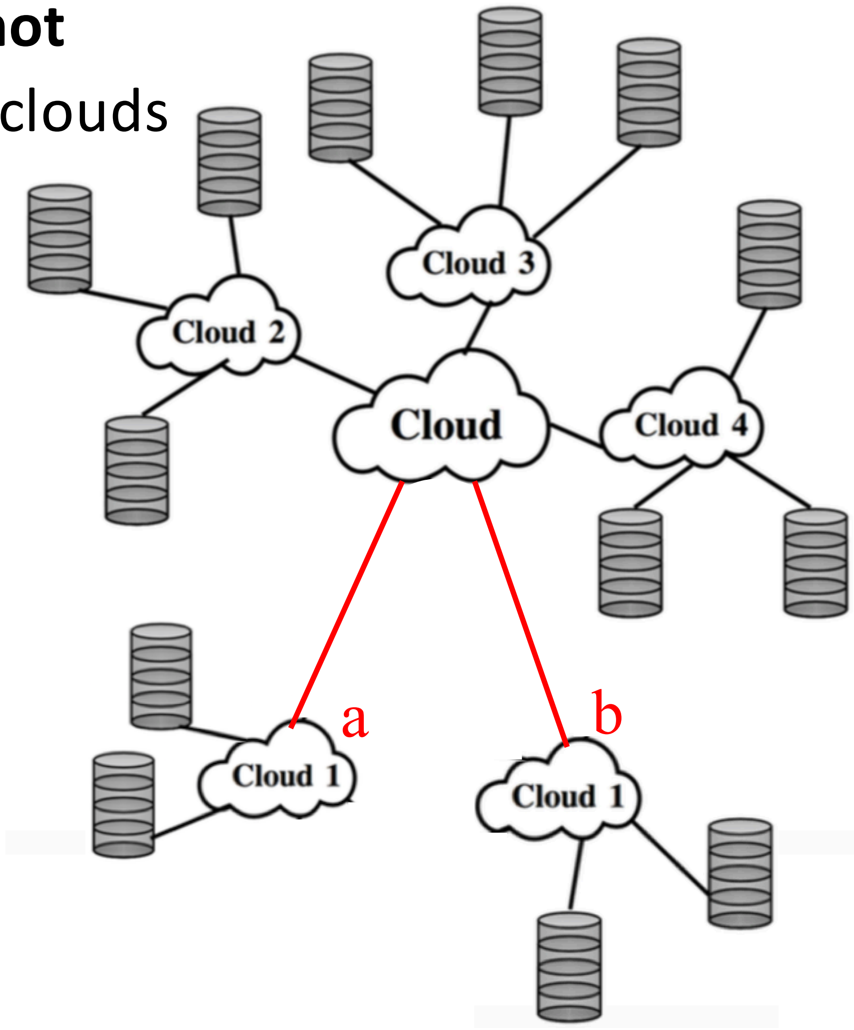
Flexibility

- The data of a local cloud become hot



Flexibility

- **The data of a local cloud become hot**
  - Split the cloud into two smaller clouds



Flexibility

- **The data of a local cloud become hot**
  - Split the cloud into two smaller clouds

$$\mathbf{G} = \left[\begin{array}{c|c|c|c|c|c|c} \mathbf{I}_{k_1} & \mathbf{A}_{1,1} & \mathbf{0} & \mathbf{A}_{1,2} & \dots & \mathbf{0} & \mathbf{A}_{1,p} \\ \hline \mathbf{0} & \mathbf{A}_{2,1} & \mathbf{I}_{k_2} & \mathbf{A}_{2,2} & \dots & \mathbf{0} & \mathbf{A}_{2,p} \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \mathbf{0} & \mathbf{A}_{p,1} & \mathbf{0} & \mathbf{A}_{p,2} & \dots & \mathbf{I}_{k_p} & \mathbf{A}_{p,p} \end{array}\right].$$

$$\mathbf{T}_x = \left[\begin{array}{c|c|c|c} \mathbf{A}_{x,x} & \mathbf{B}_{x,1} & \dots & \mathbf{B}_{x,p} \\ \hline \mathbf{U}_x & \multicolumn{3}{c}{\mathbf{Z}_x} \end{array}\right]$$

Flexibility

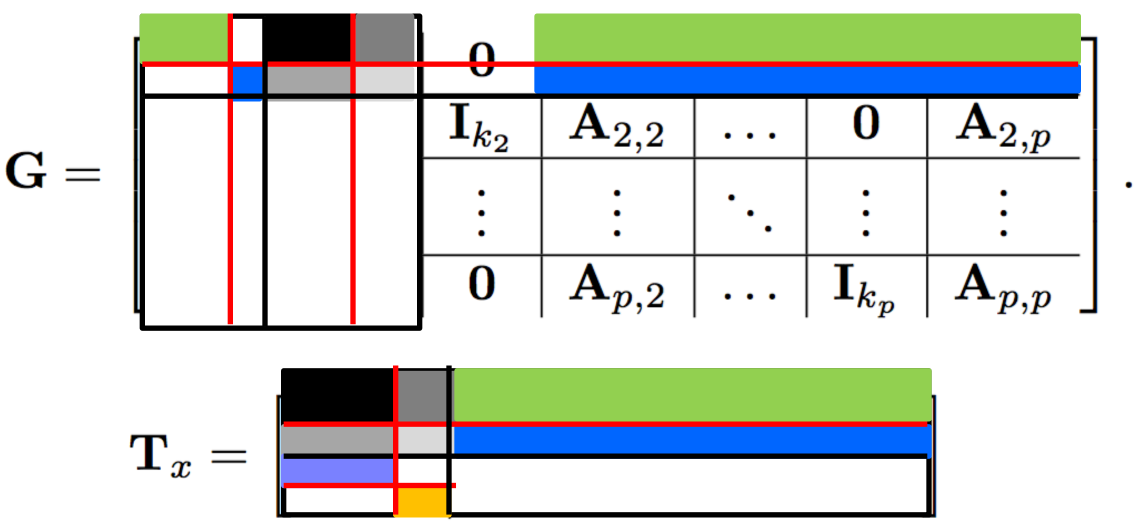
- **The data of a local cloud become hot**
  - Split the cloud into two smaller clouds
    - Divide matrices $\mathbf{G}$ and $\mathbf{T}_x$ into blocks

$$\mathbf{G} = \left[\begin{array}{c|c|c|c|c|c|c} \mathbf{I}_{k_1} & \mathbf{A}_{1,1} & \mathbf{0} & \mathbf{A}_{1,2} & \cdots & \mathbf{0} & \mathbf{A}_{1,p} \\ \hline \mathbf{0} & \mathbf{A}_{2,1} & \mathbf{I}_{k_2} & \mathbf{A}_{2,2} & \cdots & \mathbf{0} & \mathbf{A}_{2,p} \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \mathbf{0} & \mathbf{A}_{p,1} & \mathbf{0} & \mathbf{A}_{p,2} & \cdots & \mathbf{I}_{k_p} & \mathbf{A}_{p,p} \end{array}\right].$$

$$\mathbf{T}_x = \left[\begin{array}{c|c|c|c} \mathbf{A}_{x,x} & \mathbf{B}_{x,1} & \cdots & \mathbf{B}_{x,p} \\ \hline \mathbf{U}_x & \multicolumn{3}{c}{\mathbf{Z}_x} \end{array}\right]$$

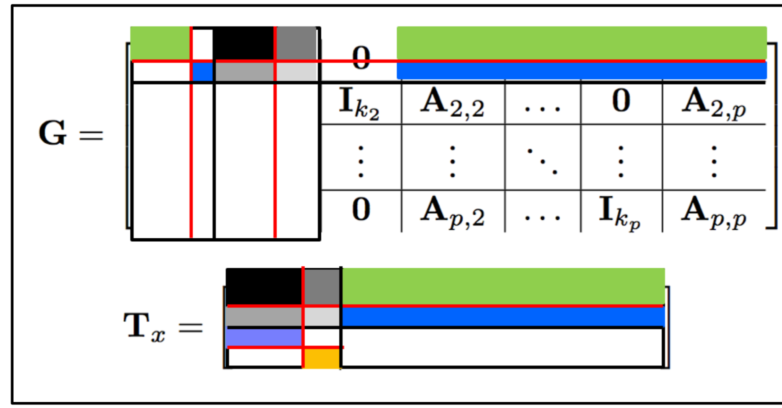
Flexibility

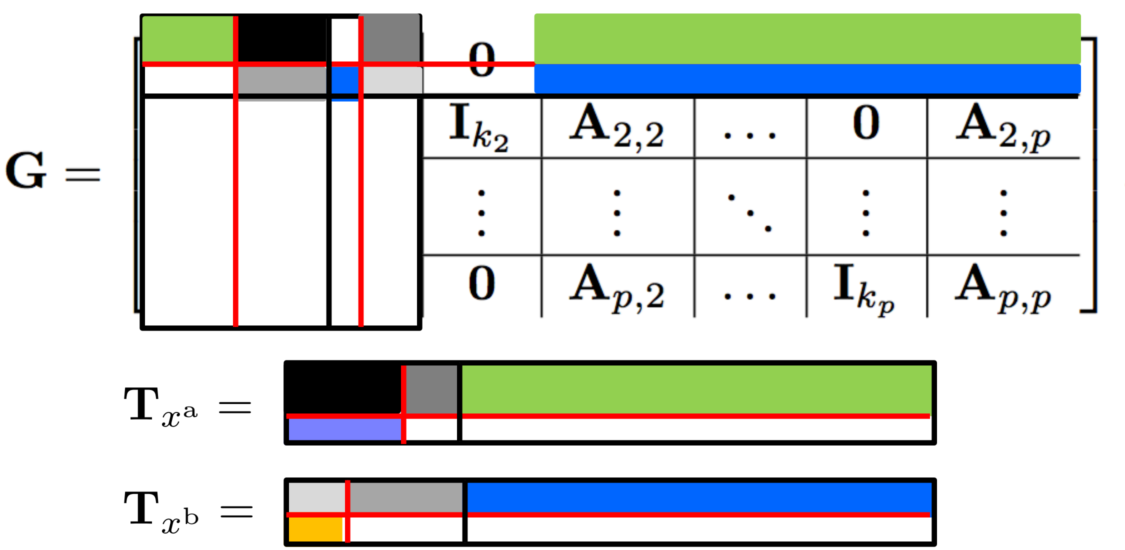
- **The data of a local cloud become hot**
  - Split the cloud into two smaller clouds
    - Divide matrices $\mathbf{G}$ and $\mathbf{T}_x$ into blocks
    - Reorder the blocks



Flexibility

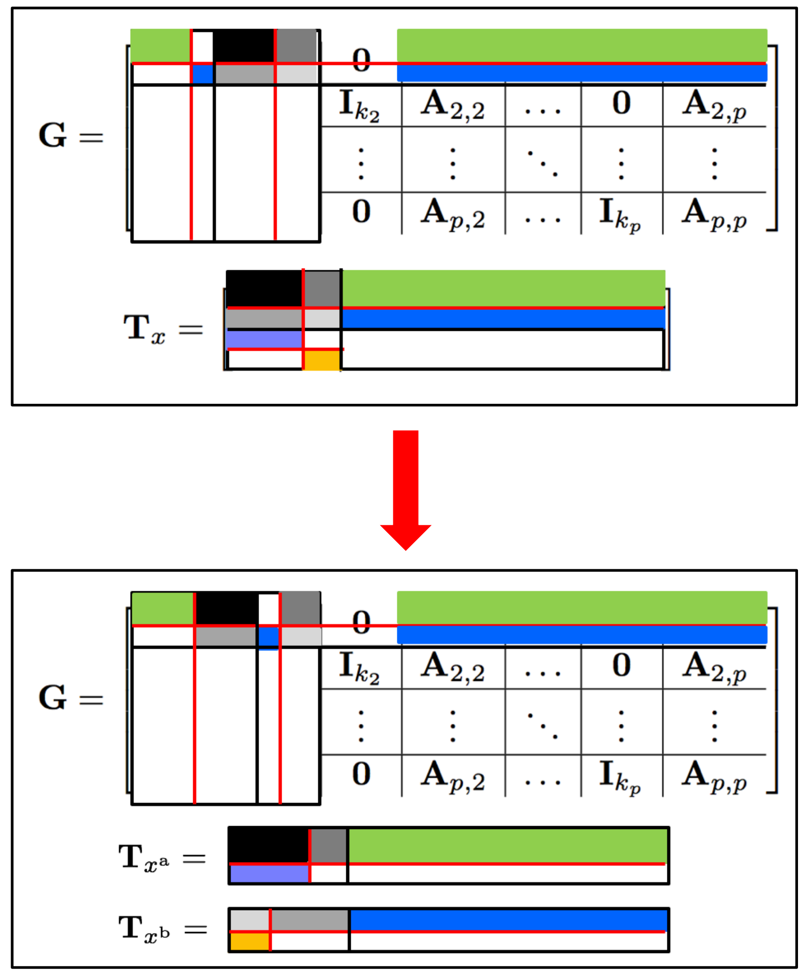
- **The data of a local cloud become hot**
  - Split the cloud into two smaller clouds
    - Divide matrices $\mathbf{G}$ and $\mathbf{T}_\chi$ into blocks
    - Reorder the blocks and obtain new matrices $\mathbf{T}_{\chi^\mathrm{a}}$, $\mathbf{T}_{\chi^\mathrm{b}}$, $\mathbf{G}$





Flexibility

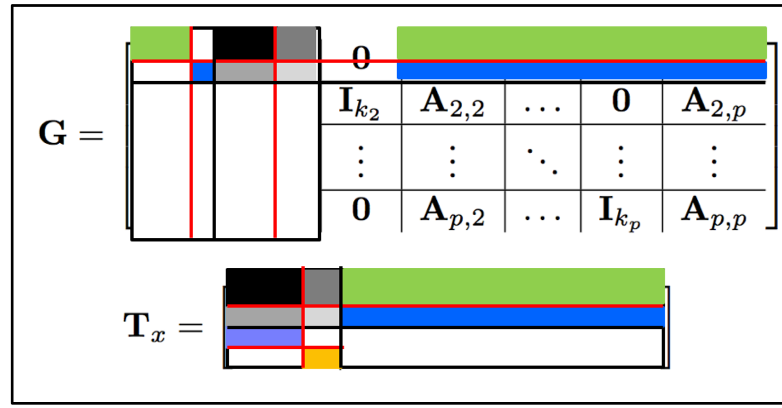
- **The data of a local cloud become hot**
  - Split the cloud into two smaller clouds
    - Divide matrices $\mathbf{G}$ and $\mathbf{T}_\chi$ into blocks
    - Reorder the blocks and obtain new matrices $\mathbf{T}_{\chi^\mathrm{a}}$, $\mathbf{T}_{\chi^\mathrm{b}}$, $\mathbf{G}$
    - **Other clouds remain unchanged**
  - Erasure correction capability
    - **Old clouds: remains unchanged**

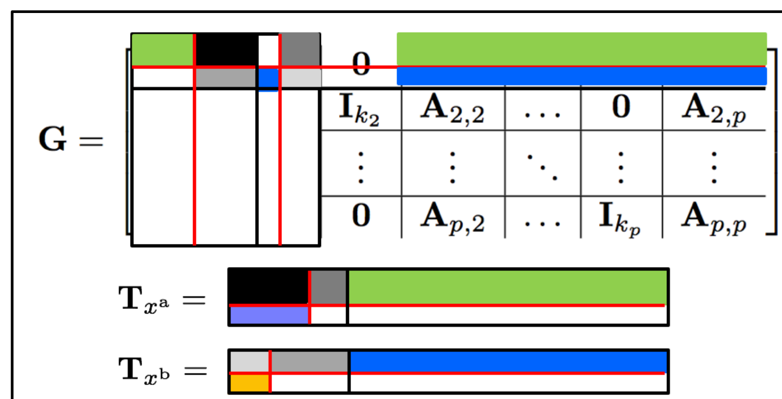


$$\begin{aligned}
\mathbf{A}_{1^\mathrm{a},1^\mathrm{a}} &= \mathbf{A}_{1,1}\left[1:k_1^\mathrm{a}, 1:r_1^\mathrm{a}\right], \\
\mathbf{B}_{1^\mathrm{b},1^\mathrm{a}} &= \mathbf{A}_{1,1}\left[k_1^\mathrm{a}+1:k_1, 1:\delta_1^\mathrm{a}\right], \\
\mathbf{A}_{1^\mathrm{b},1^\mathrm{b}} &= \mathbf{A}_{1,1}\left[k_1^\mathrm{a}+1:k_1,, r_1^\mathrm{a}+1:r_1\right], \\
\mathbf{B}_{1^\mathrm{a},1^\mathrm{b}} &= \mathbf{A}_{1,1}\left[1:k_1^\mathrm{a}, r_1^\mathrm{a}+1:r_1^\mathrm{a}+\delta_1^\mathrm{b}\right], \\
\mathbf{U}_1^\mathrm{a} &= \mathbf{U}_1\left[1:\delta_1^\mathrm{a}, 1:r_1^\mathrm{a}\right], \\
\mathbf{U}_1^\mathrm{b} &= \mathbf{U}_1\left[\delta_1^\mathrm{a}+1:\delta_1, r_1^\mathrm{a}+1:r_1^\mathrm{b}\right];
\end{aligned}$$

Flexibility

- **The data of a local cloud become hot**
  - Split the cloud into two smaller clouds
    - Divide matrices $\mathbf{G}$ and $\mathbf{T}_x$ into blocks
    - Reorder the blocks and obtain new matrices $\mathbf{T}_{x^{\mathrm{a}}}$, $\mathbf{T}_{x^{\mathrm{b}}}$, $\mathbf{G}$
    - **Other clouds remain unchanged**
  - Erasure correction capability
    - **Old clouds: remains unchanged**
    - New clouds: $r_{x^{\mathrm{a}}} - \delta_{x^{\mathrm{a}}}, r_{x^{\mathrm{b}}} - \delta_{x^{\mathrm{b}}}$ ($r_x - \delta_x$ in total)





$$
\begin{aligned}
\mathbf{A}_{1^{\mathrm{a}},1^{\mathrm{a}}} &= \mathbf{A}_{1,1}\left[1 : k_1^{\mathrm{a}}, 1 : r_1^{\mathrm{a}}\right], \\
\mathbf{B}_{1^{\mathrm{b}},1^{\mathrm{a}}} &= \mathbf{A}_{1,1}\left[k_1^{\mathrm{a}} + 1 : k_1, 1 : \delta_1^{\mathrm{a}}\right], \\
\mathbf{A}_{1^{\mathrm{b}},1^{\mathrm{b}}} &= \mathbf{A}_{1,1}\left[k_1^{\mathrm{a}} + 1 : k_1, , r_1^{\mathrm{a}} + 1 : r_1\right], \\
\mathbf{B}_{1^{\mathrm{a}},1^{\mathrm{b}}} &= \mathbf{A}_{1,1}\left[1 : k_1^{\mathrm{a}}, r_1^{\mathrm{a}} + 1 : r_1^{\mathrm{a}} + \delta_1^{\mathrm{b}}\right], \\
\mathbf{U}_1^{\mathrm{a}} &= \mathbf{U}_1\left[1 : \delta_1^{\mathrm{a}}, 1 : r_1^{\mathrm{a}}\right], \\
\mathbf{U}_1^{\mathrm{b}} &= \mathbf{U}_1\left[\delta_1^{\mathrm{a}} + 1 : \delta_1, r_1^{\mathrm{a}} + 1 : r_1^{\mathrm{b}}\right];
\end{aligned}
$$

Outline

- **Introduction**
  - Latency-reliability trade-off in storage systems
  - Heterogeneity, scalability, and flexibility
- **Preliminaries**
  - Existing literature
  - Cauchy Reed-Solomon (CRS) codes
- **Constructions**
  - Double-level codes
  - Hierarchical codes
  - Properties
- **Conclusion**

Conclusion

- **Main contribution**
  - Proposed CRS-based codes with hierarchical locality
  - Showed that our construction achieves scalability, heterogeneity and flexibility, which are critical for practical cloud storage
  - Proved that our construction requires a field size linear to the maximum codeword length

- **Future work**
  - Extend erasure-correction to error-correction, which is useful in novel SSD solutions for multi-task-oriented applications, such as autonomous driving, where latency and reliability are both important

Q&A

Thank you!